Axisymmetrical rotation of a sand heap

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This work is a phenomenological attempt to predict the dynamical response of a sand heap due to rotation about its vertical axis. We have attempted experiments and we developed a model in order to describe the effect of the rotation on the pile's surface from a dimensionless force balance equation using Coulomb's law. We obtained good agreement between the experimental patterns and the theory, depending on the material through the solid friction angle, and we gave a plausible mechanism for the way in which the history of the pile is determined by dynamical (Froude number) and material (friction coefficient) parameters.

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I. INTRODUCTION

The study of granular media has received much attention in the literature in the past few years, due to its related intriguing phenomena, such as dilatance [1], arching [2], segregation [3–5], and fluidlike behavior [6,7] which give origin to peculiar effects not occurring in other aggregation states such as liquids and solids [8–10]. These latter phenomena make difficult any theoretical description valid over a wide range of parameters. Even with an ideal granular cohesionless medium made up of monodisperse rigid grains, the analysis presents such complexity that attempts to describe its dynamical regimes are of limited success [9].

This paper deals with the free surface deformation of dry, noncohesive granular material during an axisymmetrical vertical rotation (parallel to gravity force). This phenomenon is interesting because the dynamical response of these materials composed of dense collections of solid grains are not well understood. In order to study this problem, we attempted a theoretical approach using a force balance equation that includes Coulomb's law and compared both the theoretical and experimental results, showing a good agreement between them. Experiments were made with granular material such as Ottawa sand, made up of round grains that are more or less uniformly sized. Though our experiments were made using nearly two-dimensional (2D) bins, they showed interesting properties that deviate from those found in Newtonian fluids. We obtained multivalued steady-state solutions in terms of the material parameter μ (the friction coefficient) and the Froude number, Fr [11–13], to be defined later. These solutions showed strong differences in comparison with a liquid. In particular, the analysis gives different steady-state solutions (hysteresis) for a given value of the Froude number if we reach it slowly coming up or going down. In Sec. II we describe the two-dimensional experimental setup and results. In Sec. III we present the theoretical analysis based on Coulomb's law. Comparison of experimental and theoretical results are shown in Sec. IV. Finally, in Sec. V we present final remarks and conclusions.

II. EXPERIMENTS

In order to observe and characterize the surface patterns during rotation, we did experiments with sand and obtained the free surface profiles for different motion states. Figure 1 shows a schematic frontal view of the experimental setup. Due to 3D visualization difficulties of surfaces in circular cylinders, we used thin rectangular bins with the following dimensions: 30 cm length ($R=15$ cm), 0.4 cm width, and 30 cm height. We employed granular material composed of Ottawa sand with a mean grain size between 0.03 and 0.06 cm, mean value of solid friction angle $\phi_c=31°$, and friction coefficient $\mu=\tan\phi_c=0.53$. The experiments were recorded using a videocamera with 500 frames per second and the experimental results were obtained with a high-resolution TV monitor.

The bins were filled with granular material up to $H=14$ cm height and rotated on the *z axis* (parallel to the gravity vector) while slowly changing the angular ve-

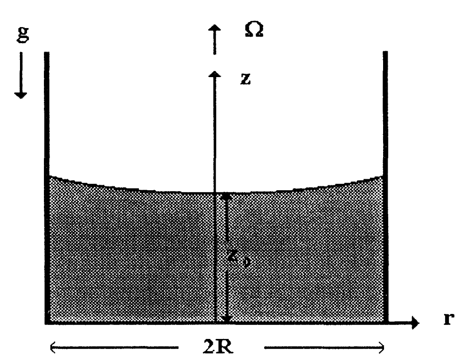

FIG. 1. Geometry of the system.

locities Ω, taking care not to introduce additional forces. The angular velocity was varied from 0 up to 52.78 s^{-1}. The resulting Froude number Fr, relating the centrifugal to gravity forces, is defined as

$$\mathrm{Fr}=\frac{\Omega^2 R}{g}\ , \tag{1}$$

where g is the magnitude of gravity acceleration. Therefore, in the experiments we use a Froude number such as $0<\mathrm{Fr}<43$. Three-dimensional effects were not observed in the thin bins (there was not any important kind of deformation in the azimuthal φ direction), and therefore the heaps can be treated as two dimensional. The width of the bin related to the particle size is about ten, excluding any effect of the sidewalls. We found several interesting patterns as we increased the Froude number, starting from zero: A first pattern, which conserves the initial free surface, occurs between $0<\mathrm{Fr}<1.36$. As the Froude number is increased further, the free surface changes, generating a peak at the center. This peak finally vanishes as the Froude number reaches a value of approximately $\mathrm{Fr}\approx 26.14$. This shape is maintained for larger Froude numbers with increasing slopes.

Another pattern can be obtained when we slowly reduce the Froude number from the maximum value reached, $\mathrm{Fr}\approx 43$. In this case, the surface shapes are clearly different than those obtained when we increased the Froude number. The final state is reached when $\mathrm{Fr}=0$, showing the free surface profiles represented by two symmetrical straight lines to an angle of 31° with respect to the horizontal. All shapes obtained are reproducible, showing a hysteresis behavior. For a given value of the Froude number we obtained different shapes depending on whether we arrive at this number coming up or going down. Otherwise, when the rotation device was not well fixed, vibrations appeared at large Froude numbers and a hole or crater was formed on the heap. With care, we can avoid this unwanted phenomenon. In Sec. IV, we present some experimental results which we compare with the theory developed in Sec. III.

III. THEORETICAL ANALYSIS

A common mechanical state of a cohesionless granular material is called the quasistatic regime, where low shear rates and high concentrations dominate. An example of this is a sand pile whose free surface forms a plane with a constant slope with the horizontal: this one is subject to shear stress (τ) and normal stress (N) that tends to move the surface's pile in accordance with Coulomb's law [14] established two centuries ago:

$$|\tau|=N\mu|\alpha|\le N\mu=N\tan\phi_c\ . \tag{2}$$

This formula expresses that the slope does not change if the shear stress is less than the product of the normal stress and the friction coefficient μ. When the yield condition (equality) is reached, that is, $\alpha=\pm 1$, the pile's surface yields, and a granular flow occurs [15]. This situation is called the critical state in soil mechanics [16]. Relation (2) has been used in studying the stability of slopes [15–20], but it also can be used here to explain the problem of rotation of cohesionless granular piles in containers.

The criterion that supports the use of Eq. (2) is the continuum point of view of granular media. In this case we can formulate a balance of forces equation for a small element of volume with density ρ at the free surface of the granular pile. Using cylindrical coordinates, this equation can be written as

$$\rho[\Omega^2 r\cos\theta-g\sin\theta]=\rho[\Omega^2 r\sin\theta+g\cos\theta]\mu\alpha\ , \tag{3}$$

where θ corresponds to the angle related to the horizontal of the free surface. The value of α can be $-1\le\alpha\le 1$, depending on the direction of the friction force, the Froude number, and the history of how we reached this value together with the initial conditions. Rearranging terms, scaling the coordinates (z and r) with the radius R of a cylindrical or rectangular container (as in the previous experiments), and introducing the Froude number Fr defined in the preceding section, we obtain from Eq. (2) a dimensionless differential equation for the surface slope as

$$\tan\theta=\frac{dz}{dr}=\frac{\mathrm{Fr}r-\mu\alpha}{1+\mu\alpha\,\mathrm{Fr}r}\ . \tag{4}$$

Assuming we slowly increase the Froude number from zero, there is a critical value of the Froude number $\mathrm{Fr}^+=\mu$ below which the surface does not show any deformation. The superscript plus sign in the Froude number indicates that the motion state results for increasing Fr. As the Froude number increases, the critical state is obtained automatically ($\alpha=1$) for $r\ge\mu/\mathrm{Fr}^+$ due to the centrifugal force. However, this critical state diffuses towards the center and can be achieved also in regions $r_c\le r\le\mu/\mathrm{Fr}^+$ due to microavalanches occurring in order to replace the granular material removed outwards in the critical region due to the centrifugal force. This process of replacing material is gradual if the Froude number is varied slowly and the preferred state for the sand is always critical, that is $\alpha=1$, for $r>r_c$. The value of r_c can be obtained by using the overall mass conservation, to be introduced later in this section. In the noncritical region $0\le r<r_c$, the surface remains with the initial shape, which in this case is horizontal, with a value of the friction parameter $\alpha=r$. Therefore, the value of α jumps from r_c to 1 at $r=r_c$ and can be given in the whole range as

$$\alpha=r+H(r-r_c)(1-r)\ , \tag{5}$$

where $H(\xi)$ represents the Heaviside function. Equation (4) with $\alpha=1$ can be integrated through the critical region $r_c\le r\le 1$ giving

$$z(r)-z_c=\frac{1}{\mu}(r-r_c)-\frac{1+\mu^2}{\mu^2\mathrm{Fr}}\ln\left[\frac{1+\mu\,\mathrm{Fr}r}{1+\mu\,\mathrm{Fr}r_c}\right]\ , \tag{6}$$

Here, z_c corresponds to the value of z at $r=r_c$. For the subcritical region $0\le r<r_c$ we obtain $z=z_0=H/R$. The maximum slope of the free surface can be obtained in the limit of a very large Froude number from Eq. (6), resulting in

$$\left.\frac{dz}{dr}\right|_{\max} = \frac{1}{\mu} \ .$$

Overall mass conservation of granular material assuming an incompressible medium can be written as

$$\int_{r_c}^{1} [z(r) - z_0] dr = 0 \ . \tag{7}$$

Introducing Eq. (6) into Eq. (7), we obtain

$$\begin{aligned} I = (z_c - z_0)(1 - r_c) + \frac{1}{2\mu}(1 - r_c)^2 \\ - \frac{1+\mu^2}{\mathrm{Fr}^{2+}\mu^3}[(1 + \mathrm{Fr}^+\mu)\ln(1 + \mathrm{Fr}^+\mu) \\ -(1 + \mathrm{Fr}^+\mu r_c)\ln(1 + \mathrm{Fr}^+\mu r_c) \\ -\mathrm{Fr}^+\mu(1 - r_c)] = 0 \ . \end{aligned} \tag{8}$$

Equation (8) gives a relationship of the form

$$I(z_c, r_c, \mathrm{Fr}^+) = 0 \ .$$

For a Froude number such as $\mu \le \mathrm{Fr}^+ < \mathrm{Fr}_1^+$, there exists an unperturbed subcritical region, resulting in $z_c = z_0$, with $r_c(\mathrm{Fr}^+)$ deduced from Eq. (8). There is a critical value of the Froude number Fr_1^+ that makes the whole region critical, that is, $r_c = 0$, generating a peak at the center with slopes equal to $-\mu$. This critical value is obtained from Eq. (8) resulting in a transcendental equation of the form $I(z_0, 0, \mathrm{Fr}_1^+) = 0$. An asymptotic relationship gives $\mathrm{Fr}_1^+ \sim 3\mu$, for $\mu \to 0$. In our case we obtain $\mathrm{Fr}_1^+ = 1.7918$ for $\mu = 0.53$. Increasing further the Froude number, the value of z_c and the height of the peak decreases, the latter vanishing in the limit $\mathrm{Fr}^+ \to \infty$. In the limiting case of $\mathrm{Fr}^+ \to \infty$, z_c reaches a minimum value of $z_{c\min} = z_0 - 1/(2\mu)$, and the surface is represented by the straight lines with a slope of μ^{-1}. The surface profiles show a minimum at $r = r_m = \mu/\mathrm{Fr}^+ > r_c$. Practically, the peak disappears as r_m becomes of the order of magnitude of the nondimensional particle size, that is, $\mathrm{Fr}^+ = O(\mu R/d)$, where d corresponds to the particle diameter. Figure 2 shows the height of the pile's center z_c as a function of the Froude number.

On the other hand, after reaching a maximum value of Fr^+, $\mathrm{Fr}_{\max}^+$, we decrease the Froude number slowly, reaching again the zero Froude number. In this case, there exist two different regions. The critical conditions now occur at the center of the bin, diffusing outward as the Froude number Fr^- decreases. This critical state is reached by microavalanches produced because the centrifugal force is unable to support the grains at the surface. In the now subcritical region $r > r_c$, the slope of the surface does not change from the value obtained for the maximum Froude number. In the critical region, the slope of the surface can be obtained from Eq. (4), but with a value of $\alpha = -1$. The position of r_c can be obtained by equating the slopes from Eq. (4) as follows:

$$\frac{\mathrm{Fr}^- r_c + \mu}{1 - \mu\,\mathrm{Fr}^- r_c} = \frac{\mathrm{Fr}_{\max}^+ r_c - \mu}{1 + \mu\,\mathrm{Fr}_{\max}^+ r_c} \ , \tag{9}$$

giving a quadratic equation for r_c. For finite values of $\mathrm{Fr}_{\max}^+$, we obtain two values of r_c for a given value of Fr^-. Criticality is reached automatically in the region $r_{c1} < r < r_{c2}$. However, the central core achieves also the critical state due to microavalanches from the critical region. Therefore, the critical region goes from $r = 0$ to $r = r_c = r_{c2}$. From Eq. (9) we can obtain the Froude number as a function of r_c as

$$\mathrm{Fr}^- = \frac{\mathrm{Fr}_{\max}^+ r_c(1-\mu^2) - 2\mu}{2\,\mathrm{Fr}_{\max}^+ r_c^2 \mu + r_c(1-\mu^2)} \ . \tag{10}$$

In the limit $\mathrm{Fr}_{\max}^+ \to \infty$, Eq. (10) can be reduced to give r_c as a function of the actual Froude number

$$r_c = \frac{1-\mu^2}{2\mu\,\mathrm{Fr}^-} \ . \tag{11}$$

Figure 3 shows the value of r_c as a function of Fr^- for different values of $\mathrm{Fr}_{\max}^+$. For $r > r_c$, the profile is the same as that obtained with the maximum Froude number. For $r < r_c$ the surface equation can be obtained from Eq. (4) with $\alpha = -1$, resulting in

$$z(r) - z_c = \frac{1}{\mu}(r_c - r) - \frac{1+\mu^2}{\mu^2 \mathrm{Fr}^-} \ln\left[\frac{1-\mu\,\mathrm{Fr}^- r}{1-\mu\,\mathrm{Fr}^- r_c}\right] \ , \tag{12}$$

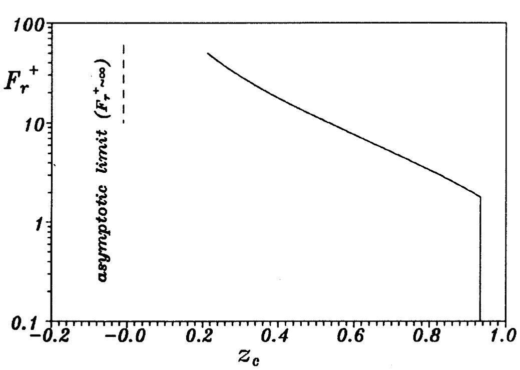

FIG. 2. Critical region height for increasing Froude numbers.

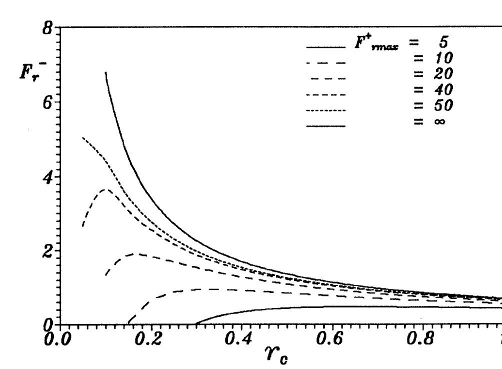

FIG. 3 Critical region radius as a function of the actual Froude number Fr^- for different values of $\mathrm{Fr}_{\max}^+$.

where z_c can be obtained in the same form as before, using the overall mass conservation. Therefore, for the same Froude number reached from both sides, we obtain in this case two different surface equations (i.e., same as found experimentally). In general, there will be an infinite number of possibilities, depending on the history, of how we arrived at that given Froude number, showing the nonlinear character of the problem.

From Eq. (4) we recover also the Newtonian fluid behavior in the case of $\mu=0$ and $\mathrm{Fr}\neq 0$, the solution of which can be given in dimensionless form as

$$z-z_0=\frac{\mathrm{Fr}}{2}r^2 . \tag{13}$$

IV. COMPARISON BETWEEN THEORY AND EXPERIMENTS

On the basis of the analysis, we can show some shapes of the surface of piles resulting from rotation. Assuming we start the motion from rest with an initial flat horizontal surface, we obtain a peak at the center with decreasing height as the Froude number increases. Figure 4 shows two-dimensional projections of surfaces generated by slowly increasing Fr^+. The lines show the theoretical results while the symbols represent the experimental data. We took a value of $\mu=0.53$ ($\phi_c=31°$) in order to compare the theory with the experiments made with Ottawa sand. The values of the chosen Froude number were $\mathrm{Fr}^+=4.0$, where a clear central peak is noted, $\mathrm{Fr}^+=26.14$, and $\mathrm{Fr}^+=\mathrm{Fr}^+_{\max}=42.57$.

On the other hand, if we decrease the Froude number from $\mathrm{Fr}^+_{\max}=42.57$, we obtain another type of solution for the surface equation. Figure 5 shows the solutions for $\mathrm{Fr}^-=26.16$, $\mathrm{Fr}^-=4.0$, and finally, for $\mathrm{Fr}^-=0$, where we obtain the final state for the surface with a constant slope $\mu(\phi_c=31°)$. In all cases presented here, there is good qualitative agreement between theory and experiment, which confirms that the present model describes correctly the phenomenology of the experiment. The experimental values of the surface profiles are found to be slightly below the theoretical ones. This is due to the packing of the grains not considered in the analysis.

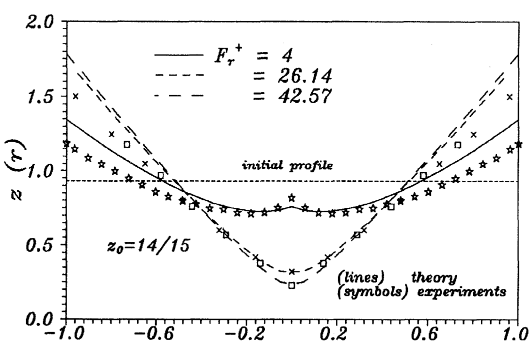

FIG. 4. Free surface profiles obtained from the analysis (lines) and from experiments (symbols) as the Froude number increases, for $\mathrm{Fr}^+=4,26.14$, and 42.57.

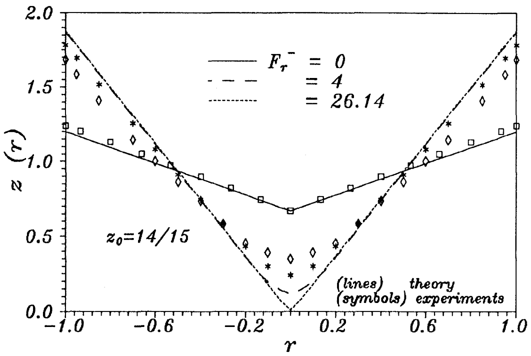

FIG. 5. Free surface profiles obtained from the analysis (lines) and from experiments (symbols) as the Froude number decreases, for $\mathrm{Fr}^-=0$, 4, and 26.14.

We should comment that the friction angle actually does not have a unique value. It fluctuates within a small range [21], which in our experiments was $\phi_c\pm\delta$, where $\delta\sim 1°$. For each Froude number, the experimental results deviate a few percent (less than 2%) in the surface shape profiles.

V. REMARKS AND CONCLUSIONS

The problem of the rotation of granular material on the vertical axis, in general, is a very complex phenomenon because it takes into account not only the gravitational as well as the centrifugal forces but also the history of the motion through the friction force. However, the history or memory effect disappears for continuously slowly increasing or decreasing rotation, as the grain achieves the critical state everywhere. In this case, from a continuum point of view, this problem can be understood and a simple analysis can describe correctly the motional behavior. Even in the case of slowly changing Froude number, the system shows hysteresis, indicating the existence of multiple steady-state solutions. In the more general case of reaching a given value of the Froude number through any arbitrary way (rapid step type changes), it is possible to obtain any number of steady-state solutions. Hysteresis in avalanche processes is related to the changes in the slope near the maximum angle and the frictional and packing factors inside the piles. In the problem described in this work the hysteresis behavior is related to these factors but additionally is most related to the initial and boundary conditions.

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