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Hysteresis in Granular Media Subjected to Axisymmetrical Rotation

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In this work we have studied the hysteretic behavior for the mean slope of the free surface of noncohesive granular media under axisymmetrical rotation. In order to study the hysteresis loop the relationship between the average slope of the free surface (most important measurable quantity) and the Froude number has been established. Differences between experiments and theoretical predictions are also shown. Theory predicts that hysteresis must disappear as the friction coefficient μ of the granular material reaches unity. However, this condition could not be reached experimentally using tapioca (where $\mu \sim 1$), due to the important difference between the angle of internal friction and the angle of repose for small containers.

KEYWORDS: mechanics of discrete systems, granular matter, free surface flow

§1. Introduction

Hysteresis is very common phenomenon in many granular systems.[1-3] Its main cause is due to the changing direction of the friction force between grains and also due to the difference between static and dynamic friction forces. In the particular case of axisymmetrical rotation of granular material,[4,5] hysteresis is also present. In a previous paper,[5] we studied the free surface deformation of dry, non cohesive granular material during an axisymmetrical vertical rotation (parallel to gravity force). In order to study this problem, we did a theoretical approach using a force balance equation which includes the Coulomb yield condition and compared both the theoretical and experimental results, showing a good agreement between them. Experiments were made in a thin rectangular box of half-length $R = 15$ cm, partially filled with granular material like Ottawa sand, made up of round grains that are more or less uniformly sized. We obtained multi-valued steady state solutions in terms of the material parameter μ (the friction coefficient) and the Froude number, F defined as the ratio of the centrifugal to gravity forces, $F = \Omega^2 R/g$, where Ω corresponds to the angular velocity and g corresponding to the gravity acceleration. The analysis gave different steady-state solutions (hysteresis) for a given value of the Froude number if we reach it slowly coming up or down. The analysis shows that the hysteresis (after forgetting the initial surface profile) disappears as the value of the friction coefficient μ approaches unity.

The objective of this paper is to study systems with different values of the friction coefficient, including the case of $\mu \to 1$, and compare the results with experimental results. This comparison is made in terms of the average slope of the free surface as a function of the Froude number. The averaged slope is representative of the free surface profile and is a simple experimental measurable quantity. The experiments permit us to deduce the very important role of the angle of repose on the hysteresis loop.

§2. Analysis

In this Section we give a brief description of the theoretical analysis developed in ref. 5 and extended to the case of several cycles produced by increasing and later decreasing the Froude number values to obtain a steady hysteresis loop for the free surface deformation in the experimental set-up explained in the previous section. We employ the superscript plus sign on the Froude number, F^+, to indicate that the motion state results from the increasing the angular velocity. Similarly, the superscript minus sign on the Froude number, F^-, corresponds to the motion state resulting from decreasing the angular velocity. We use the Coulomb's yield condition, which gives a relationship between the shear force (τ) and the normal force (N) on the near free surface, in the form

$$|\tau| = N\mu|\alpha| \le N\mu = N\tan\phi, \tag{1}$$

where $\mu = \tan\phi$ and ϕ is the angle of internal friction. If the yield condition (equality) is reached, that is $|\alpha| = 1$, the pile's surface yields and thus a granular flow takes place.

Using the coordinates r and z for the horizontal and vertical directions, respectively, the force balance equation of a small element of volume at the free surface of the granular pile, let us to obtain a first order nondimensional differential equation for the free surface as

$$\frac{dz}{dr} = \frac{Fr - \mu\alpha}{1 + \mu\alpha Fr}. \tag{2}$$

We scaled the coordinates (r, z) with the half-length of the rectangular bin, R. The value of α can be $-1 \le \alpha \le 1$, depending on the direction of the friction force, the Froude number, the initial conditions and the history how we reached this value. Integrating eq. (2) for increasing values of the Froude number, F^+, with $\alpha = 1$, we obtain for relatively large values of F^+[5]

$$z(r) - z(0) = \frac{r}{\mu} - \frac{1 + \mu^2 \ln(1 + \mu F^+ r)}{\mu^2 F^+}. \quad (3)$$

This equation can be transformed to

$$z(r) - z(0) = \overline{\frac{dz}{dr}} r - g(r). \quad (4)$$

where $\overline{dz/dr}$ represents the averaged nondimensional slope as the Froude number increases and is given by

$$\overline{\frac{dz}{dr}} = \int_0^1 \frac{dz}{dr} dr \sim \frac{1}{\mu} - \frac{1 + \mu^2}{\mu^2} \frac{\ln(1 + \mu F^+)}{F^+}. \quad (5)$$

Function $g(r) > 0$ corresponds to the deviation from the actual shape and that given by an uniform slope

$$g(r) = \frac{1 + \mu^2}{\mu^2 F^+} \left[\ln(1 + \mu F^+ r) - r \ln(1 + \mu F^+)\right]. \quad (6)$$

Due to the fact that $g(r)$ is always positive, the root mean square (rms) of the deviation is then

$$\overline{g(r)} = \int_0^1 g(r) dr = \frac{1 + \mu^2}{\mu^2 F^+} \left[\ln(1 + \mu F^+) \left(\frac{1}{2} + \frac{1}{\mu F^+}\right) - 1\right]. \quad (7)$$

The free surface can be characterized by two parameters which are the mean slope and the rms deviation. These are plotted in Fig. 1 for $\mu = 0.53$. For a given value of the Froude number we obtain a single value of both the mean slope and the rms deviation. The mean slope is monotonic with the Froude number, F^+ and is easy to measure experimentally. The rms deviation, however, is multivalued (for a given value of the rms deviation we have two values for the Froude numbers). This is the reason why we can take the mean slope as the important parameter which characterizes the free surface shape. Therefore, the maximum nondimensional slope of the free surface can be obtained in the limit of very large Froude number from eq. (5), resulting

$$\left.\overline{\frac{dz}{dr}}\right|_{max} = Lim_{F^+ \to \infty} [z(1) - z(0)] \to \frac{1}{\mu}. \quad (8)$$

The surface is then represented by a wedge with constant slopes $\pm 1/\mu$. In the same form, the integration of eq. (2) with $\alpha = -1$, can be obtained when reducing the Froude number, F^-, from a maximum Froude number $F^+ = F^+_{max} \to \infty$, resulting the average slope given by

$$z(1) - z(0) = -\frac{1}{\mu} - \frac{1 + \mu^2}{\mu^2 F^-} \ln(1 - \mu F^-). \quad (9)$$

In the limit $F_r^- \to 0$, we obtain from eq. (9)

$$\left.\overline{\frac{dz}{dr}}\right|_{min} = Lim_{F^- \to 0} (z(1) - z(0)) \to \mu. \quad (10)$$

This last result indicates that the granular material surface shape reduces to a wedge with constant slopes given by the angle of internal friction, as the angular velocity vanishes. Therefore, the two limiting solutions for $F^+_{max} \to \infty$ and $F^- \to 0$ are characterized by wedges with constant but different slopes. The transition from one to the other limit can be studied therefore using the averaged slope of the surface profiles as the main dependent variable.

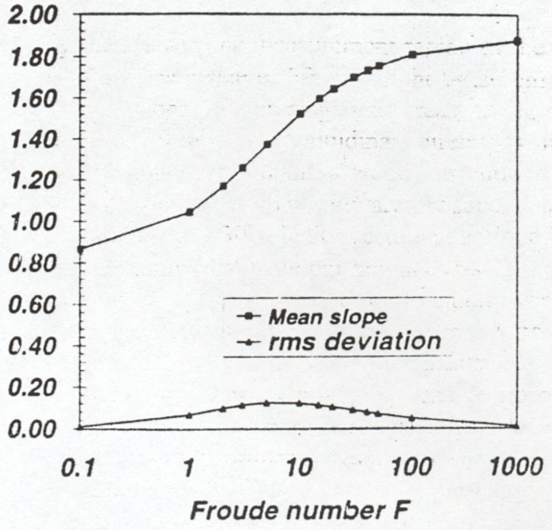

Fig. 1. Mean slope and the rms deviation as a function of the Froude number obtained when increasing the angular velocity.

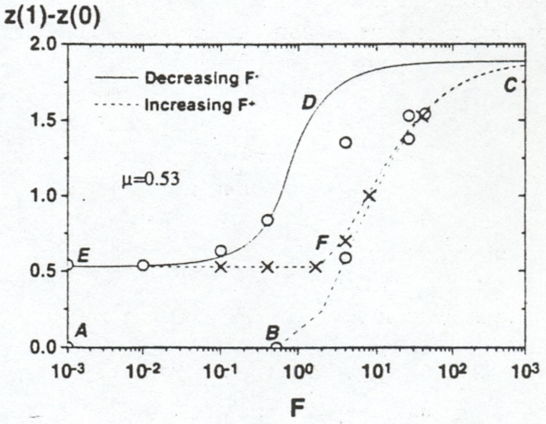

Fig. 2. In this graph we plot the average slope $z(1) - z(0)$ as a function of the Froude number for $\mu = 0.53$, corresponding to Ottawa sand. Here dashed lines corresponds to increasing Froude numbers. The symbols ○ corresponds to the increasing-decreasing part of the hysteresis loop starting at $F = 0$. Later, the mean slope reaches the angle of repose. Restarting again the motion by increasing F, gives the curve starting at point E, being characterized experimentally by symbols ×.

Figure 2 shows a typical hysteresis behavior for values of $\mu < 1$, plotting the average nondimensional surface slope as a function of the Froude number. In this plot we take $\mu = 0.53$. We plot both the analytical (continuous and broken lines) as well the experimental results (symbols), which are to be explained in the following section. We assume at the beginning a horizontal surface for the granular material (any other initial surface profile can also be easily assumed, being not important for the steady hysteresis loop). This corresponds to the point A in the graph. If we increase the Froude number, the surface remains unperturbed until a critical value of Froude number is achieved, $F^+ \approx \mu$ (B). The nondimensional height difference increases then monotonically with the Froude number, reaching the asymptotic value of $1/\mu$ (C) as $F^+ \to \infty$. At this point the initial profile has been completely forgotten. If we decrease now the Froude number, the nondimensional height difference also decreases reaching the asymptotic value of μ as $F^- \to 0$. The solution is given by the curve C-D-E. Increasing again the Froude number, there is a subcritical region, $r < r_c$, where the slope of the surface remains unchanged, that is

$$z(r) - z(0) = \mu r, \text{ for } 0 \leq r \leq r_c. \tag{11}$$

At $r = r_c$, α reaches the value of unity, $\alpha(r_c) = 1$. Introducing $\alpha = 1$, $dz/dr = \mu$, at $r = r_c$, from eq. (2), we obtain r_c as a function of the actual Froude number, F^+ as

$$r_c = \frac{2\mu}{(1-\mu^2)F^+}. \tag{12}$$

This gives a critical Froude number $F_c^+ = 2\mu/(1-\mu^2)$, below which the surface slope remains unchanged everywhere. For $r > r_c$, the surface equation is given by eq. (3). Thus, the average of the nondimensional slope for $F^+ > F_c^+$, takes the form

$$z(1) - z(0) \sim \frac{2\mu^2}{(1-\mu^2)F^+} + \frac{1}{\mu}\left(1 - \frac{2\mu}{(1-\mu^2)F^+}\right) - \frac{1+\mu^2}{\mu^2 F_r^+}\ln\left(\frac{1+\mu F^+}{1+\frac{2\mu^2}{(1-\mu^2)F^+}}\right). \tag{13}$$

For $\mu = 0.53$ the trajectory follows now the curve E-F-C, as we increase the Froude number again. The area inside the hysteresis loop reduces as the value of μ increases, vanishing for $\mu \geq 1$.

For $\mu \geq 1$. The asymptotic shape of the surface, for $F^+ \to \infty$, is an inverted cone with a slope of $1/\mu \leq 1$, which is lower than that of the angle of internal friction. Once this shape is reached, the centrifugal force is unable to change this profile, even if we reduce it completely to $F^- = 0$. In this case the hysteresis phenomenon disappears.

§3. Experiments

In order to test how the hysteresis loop decreases as we increase the friction coefficient, we have made experiments using the same technique previously employed by the authors in order to characterize the dynamical evolution of the free surface shape.[5] The noncohesive granular material used in this work was tapioca (with a friction coefficient close to unity), which is composed by rough, nearly spherical grains with 0.22 ± 0.02 cm in diameter (projected area diameter). 1.36 gr/cm^3 is the grain density and angle of internal friction $\phi = 43° \pm 1°$ ($\mu = 0.93$). Actual materials under gravity also has an angle of repose, θ_r, which is defined as the angle between the horizontal and the free surface of the heap after a land slide has restored the heap to a metastable equilibrium slope. In our experiments $\theta_r = 38°$ for tapioca and, therefore, our angles ϕ and θ_r which were measured precisely by using the experimental set-up as was proposed by,[4] and are slightly different than that obtained previously by other authors, i.e., $\phi = 45.5°$ ($\mu = 1.01$) and $\theta_r = 37.5°$.[6,7] For comparison, we also have employed

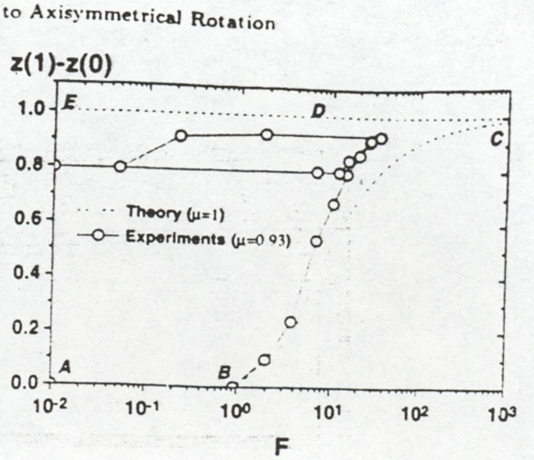

Fig. 3. Hysteresis loop for the ideal case with $\mu = 1$ ($\phi = \theta_r$). The curves C-D-E and E-F-C in this case are identical and therefore the point F does not appear in the graph. The experimental data with $\mu = 0.93$ are represented by symbols ○. However, here the material has $\phi \neq \theta_r$ and the corresponding difference at $F^- = 0$ is noted.

granular material composed of Ottawa sand with mean grain's size between $0.03 - 0.06$ cm, $\phi = 28°$ ($\mu = 0.53$) and $\theta_r = 27°$.

Experiments were made using a near 2D bin, filling by raining down the grains up to $H = 14$ cm. For sand the bin has had the next dimensions: 1 cm width, 30 cm height, and 30 cm length and for tapioca we only changed the width to 2 cm. This bin was rotated within the range $0 \leq F \leq 43$ and the dimensionless average slope, $z(1) - z(0)$, was carefully measured for tapioca and Ottawa sand and several values of F. Important differences for both cases were noted.

In Fig. 2 we also have plotted the experimental mean slope (symbols) for Ottawa sand. The data fits the theoretical curve except that obtained from reducing the Froude number $F^- \to 0$ (curve C-D-E), because the theoretical asymptotic limit $F^+ \to \infty$, could not be reached experimentally. If the system could reach the limit $F^+ \to \infty$, then the fit between data and theory would be excellent. The mean slope reached for $F^- = 0$, agrees well with the predicted value, $\mu = 0.53$, because in this case $\phi \approx \theta_r$.

A different situation occurs using tapioca, with $\mu \doteq 0.93 \sim 1$, but with $\phi - \theta_r = 5°$. In this case the surface averaged slope did not reach the value $z(1) - z(0) = \mu \sim 0.93$ as $F^- = 0$. We obtain instead the value $z(1) - z(0) \sim 0.8$, which is very close to $\tan\theta_r$. This difference is clearly noted in Fig. 3 where the experimental hysteresis loop (symbols) is qualitatively similar to that using sand (Fig. 2). The area inside the loop is clearly smaller than that obtained with $\mu = 0.53$. For comparison we have plotted in the Fig. 3, the hysteresis cycle for the ideal case with $\mu = 1$ and without dilatance ($\phi = \theta_r$). In this latter, the curves C-D-E and E-F-C would be identical and therefore the slope would be always constant, independent of the Froude number.

§4. Conclusions

We studied in this paper the effect of the friction coefficient μ and the angle of repose θ_r of the granular material on the hysteretic behavior of the free surface shape under axisymmetrical rotation. The analysis shows that for granular media with $\mu \geq 1$ and $\theta_r \geq 45°$, the steady hysteresis loop, produced by increasing and decreasing the Froude number, disappears completely. In this hypothetical case the surface shape (inverted cone with slope of $1/\mu$) remains unperturbed, being independent of the Froude number.

We used a material called tapioca in our experimental work, with a measured coefficient of $\mu \simeq 0.93$ and an angle of repose of $\theta_r \simeq 38°$. We were unable to find a granular material with a larger friction coefficient. We showed that hysteresis decreases in strength as the value of the friction coefficient increases. We also found important deviations in relation to the ideal case, introduced by the angle of repose which has their origin in the dilatance phenomenon. This last phenomenon vanished for $F \to \infty$, by increasing R towards infinite at finite Ω. In this case, in fact, $\phi = \theta_r$ and therefore the hysteresis loop disappears.[8] In our model we have ignored the compressibility effects, which could be an important factor.[4]

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