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On a universal description for the fracture patterns in rotating cohesive granular media

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Abstract. – We discuss the existence of a universal description for the fracture patterns in a small sample of cohesive granular material under axisymmetrical rotation. This universal description can be established through a dimensionless relation between the material parameters, the Froude number, the characteristic lengths of the sample and the size of the fracture. We show that, for a xerographic developer, it is possible to plot all the experimental data given in *Powders and Grains 97*, by Genovese F. C. *et al.* ((A. A. Balkema, Rotterdam) 1997, p. 135) in a single universal line.

Introduction. – Since Coulomb's work [1], cohesive granular materials have been studied, mainly in the context of soil mechanics, where the possibility of a quasistatic deformation (yield) has important consequences. In the limit of yield, these materials should also exhibit fracture patterns which depend on the size and geometry of the granular sample and on the relative intensity and orientation of the relevant stresses. Rather, for small enough samples of granular material yield occurs, obeying Coulomb's yield criterion (CYC), along the slip planes which are planes with a definite orientation (angle of slip) but without a unique spatial localization (*i.e.*, there always exists a family of possible slip planes with the same angle of inclination) [2, 3]. Surprisingly, this indetermination in the exact localization of the fracture can be eliminated in unconfined small samples of cohesive materials subjected to extensive and adhesive stresses.

The fracture phenomenon has a critical importance in order to optimize some technological applications, such as transport of xerographic powders [4, 5]. Moreover, phenomena in these dry cohesive materials under quasistatic deformation should be treated at two levels of description: a) the detailed microstructure and the mechanical and electrostatic forces of interaction between grains are important and b) the microscopic effects must be specified into a balance of force equation through the inclusion of some phenomenological coefficients. These coefficients could be the cohesion and internal friction of the material and the adhesion (between the substrate and the granular material). In order to understand the main fracture

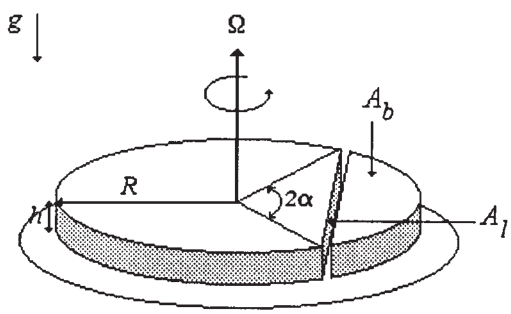

Fig. 1. – Schematic illustration of the fracture and the separated slice in a cohesive granular medium under axisymmetrical rotation of magnitude Ω and gravity acceleration g. The slice has a fracture plane of area A_l and the area of this slice in contact with the substrate is denoted by A_b.

patterns in a macroscopic sample of cohesive granular material under rotation, we will use the latter approach.

The aim of this letter is to discuss the convenience and usefulness of a universal description for the fracture patterns in cohesive granular media under axisymmetrical rotation and gravity action of a sample of cylindrical shape. Comparison with experimental data obtained from xerographic powders is also presented in order to validate our approach. In order to reach this goal, our work is divided as follows. First we describe the problem and nondimensionalize the force balance equation presented in [5]. Under this appropriate normalization, we analyze next the asymptotic limits of low and high cohesivity. In these limits we found analytical expressions for the fracture, obtaining a universal description of the problem. We show that the experimental data obtained in [5] for a xerographic developer fit very well in the single universal curve. Finally, we give a discussion of the main results and the limitations of the model.

Theoretical approach. – Recently experimental evidence has been found of rupture in well-defined segments of dry cohesive powders due to tensile load originated during axisymmetrical rotation [5]. In these experiments cylindrical cells of xerographic powders, mounted on a horizontal disc, were rotated under the gravity acceleration (g) and patterns such as the one shown in fig. 1 were obtained. The aim of such experiments was to obtain the critical minimum angular velocity, Ω, which generates a fracture in the cylindrical specimen of radius R, for a series of xerographic powders having several levels of cohesivity. The fracture angle α was also measured. In this work [5], an analysis of this process has been made, using a continuum approach. However, the theoretical analysis for high and low cohesivity can be simplified and generalized by exploring the respective asymptotic limits. We assume, as in [5], that the fracture occurs for a cylinder of (small) height h with an angle of fracture 2α, bulk density ρ, cohesion coefficient σ_t, internal friction μ and adhesion coefficient (between the substrate and the powder) σ_w. We also assume that the fracture process occurs along a vertical plane as shown in fig. 1. This is justified because it occurs very frequently in a long series of experiments [6]. However, other fracture shapes may be formed, but less frequently [6]. Regarding this point, we do not know the reason why the fracture plane is vertical in most of the cases.

The fracture phenomenon has been modeled using Coulomb's yield condition (CYC) [1], giving a relation between the tangential forces along the fracture plane τ, the normal forces to the fracture plane N (created during the fracture) and the cohesive forces C between the fractured slice and the disc. In the yield limit, the CYC given by $|\tau| \leq \mu N + C$ produces after a spatial averaging of the forces acting on the slice, the limiting form [5]

$$\rho\Omega^2 R^3 hu = \mu\left[\rho g A_b h + \sigma_w A_b\right] + \sigma_t A_l, \tag{1}$$

where $u = u(\alpha)$ is a dimensionless function represented by

$$u = 2\int_0^\alpha \cos\gamma \mathrm{d}\gamma \int_{\frac{\cos\alpha}{\cos\gamma}}^1 r^2 \mathrm{d}r = \frac{2}{3}\sin^3\alpha. \tag{2}$$

The left-hand side of eq. (1) corresponds to the centrifugal force on the fractured slice acting in a direction normal to the fracture plane. The first term on the right-hand side is the adhesion and friction forces, while the second one is due to the cohesion force. A_b and A_l are the contact area of the fractured slice with the substrate and the area of the fracture plane, respectively, and are given by

$$A_b = R^2\left(\alpha - \frac{\sin 2\alpha}{2}\right) \qquad \text{and} \qquad A_l = 2Rh\sin\alpha. \tag{3}$$

The nondimensional form of eq. (1) can be written as

$$H = Kf(\alpha) + g(\alpha), \tag{4}$$

where

$$H = \frac{\rho g R \mathrm{Fr}}{\sigma_t}, \qquad \text{with} \qquad \mathrm{Fr} = \frac{\Omega^2 R}{g}. \tag{5}$$

In eq. (4) we have assumed that $\alpha \neq 0$. Fr is the Froude number, which relates the centrifugal to gravity forces and the dimensionless parameter K has the form

$$K = \frac{\rho g R}{\sigma_t}\mu\left(1 + \frac{\sigma_w}{\rho g h}\right) = \frac{\rho g R}{\sigma_t}\mu'. \tag{6}$$

The effective coefficient of friction μ' is then

$$\mu' = \mu\left(1 + \frac{\sigma_w}{\rho g h}\right). \tag{7}$$

The functions $f(\alpha)$ and $g(\alpha)$ depend only on the fracture angle α and are given by

$$f(\alpha) = \frac{3}{2}\frac{\alpha - \frac{\sin 2\alpha}{2}}{\sin^3\alpha} \qquad \text{and} \qquad g(\alpha) = \frac{3}{\sin^2\alpha}. \tag{8}$$

By using eqs. (4) and (8), we note easily that the null-cohesivity limit, $\sigma_t = 0$, is singular, *i.e.* K and H diverge in this limit.

In the asymptotic limit of high cohesion, $\sigma_t \to \infty$, and therefore $K \ll 1$ ($\alpha \longrightarrow \pi/2$), it can be easily shown that $f(\alpha) \sim 3\pi/4$, $g(\alpha) \sim 3$, and consequently

$$H \sim \frac{3\pi}{4}K + 3, \qquad \text{for} \qquad K \to 0. \tag{9}$$

Similarly, in the limit of low cohesion, $\sigma_t \to 0$, which yields $K \gg 1$ ($\alpha \longrightarrow 0$), we obtain

$$f(\alpha) \sim 1 + 0.3\alpha^2 + 0.607\alpha^4 + O(\alpha^6) \qquad \text{and} \qquad g(\alpha) \sim \frac{3}{\alpha^2} + 1 + O(\alpha^2). \tag{10}$$

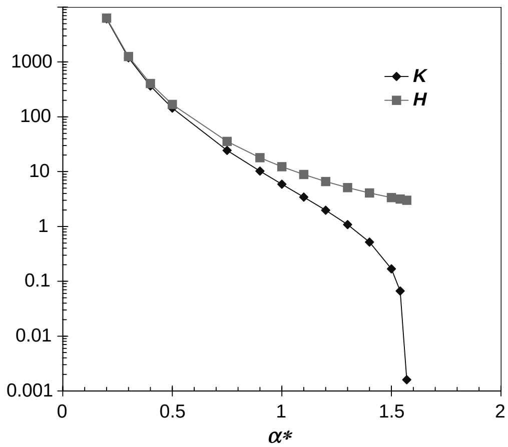

Fig. 2

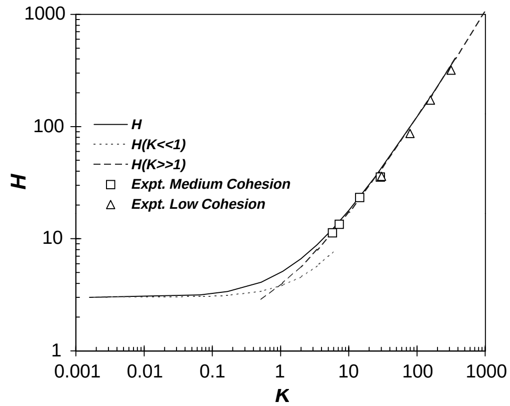

Fig. 3

Fig. 2. – Parameters H and K as a function of the fracture angle α^*.

Fig. 3. – Log-log plot of H as a function of K for a wide range of values of cohesion. The symbols correspond to experimental data [5], while the continuous line corresponds to the universal curve obtained using a numerical iteration. Dashed lines indicate the asymptotic solutions for low and high cohesivity.

The value of α which produces the minimum value of H, $\alpha = \alpha^*$, in this asymptotic limit is then

$$\alpha^* \sim \left(\frac{10}{K}\right)^{\frac{1}{4}}, \tag{11}$$

which yields

$$H \sim K + 1.8973\sqrt{K} + 1 + O(K^{-\frac{1}{2}}), \qquad \text{for} \qquad K \to \infty. \tag{12}$$

Figure 2 shows the value of K and H as a function of α^*. We obtain this plot giving an initial value of $0 < \alpha^* < \pi/2$. The value of K is then deduced by $K = -g'(\alpha^*)/f'(\alpha^*)$, where primes denote derivatives with respect to α. Finally, the value of H is obtained using eq. (4) with $\alpha = \alpha^*$. Given the material parameter K, there is a minimum value of the dynamic parameter H (minimum value of the Froude number) where material yields, producing the fracture in a slice with an angle of $2\alpha^*$. The universal behavior $H(K)$ is also plotted in fig. 3. In order to show a direct comparison between theory and experiments, we have used experimental data corresponding to a granular material referred to as a developer which is made up of a mixture of toner particles and carrier beads of size around 100 μm. This material plays an essential role in many xerographic processes. The data were obtained from experiments reported by Valverde *et al.* [4] and Genovese *et al.* [5]. They used developer samples on a substrate (circular disc of 7.52 cm diameter). The cohesive sample has radius in the range 0.2 cm $\leq R \leq$ 3.0 cm and a height of $h = 0.7$ mm. The values for the cohesion coefficient ranged from $\sigma_t = 1.1$ Pa for low cohesion to $\sigma_t = 12.6$ Pa for medium cohesion. The bulk density of the developer was $\rho \sim 3$ g/cm^3. For low cohesion the reported value of μ' was $\mu' \simeq 0.59$, while for medium cohesion $\mu' \simeq 0.62$ [4, 5]. The Froude number for these experiments varied in the range $0.54 \leq \mathrm{Fr} \leq 1.21$. In fig. 3 we plot the experimental data obtained from [5], within the ranges above referred. The data fit very well the universal curve and show the usefulness and convenience of the employed normalization. Experiments with very high cohesivity were not reported in [5]. The asymptotic limits of high and low cohesivity, obtained from eqs. (9) and (12) respectively, are plotted as a visual guide.

Remarks and conclusions. – In this letter we showed the existence of a universal curve useful for the description of the fracture of cylinders, made up of dry cohesive granular material, under axisymmetrical rotation. In order to give a comparison with experiments, we have used data obtained from fracture in xerographic powders (developers) which have important technical applications. We showed that the universal curve should actually determine the size and the relative importance of the phenomenological coefficients in this phenomenon. It is possible in this problem that the free surface deformation could have also an important effect on the fracture shape, mainly in the limit of low cohesion [7]. The inclusion of this aspect in a more detailed treatment may be interesting. Work along these lines is in progress.

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