Free surface deformation of a non cohesive granular material in a box under uniform acceleration

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In this work the problem of the deformation of the free surface of a cohesionless granular material confined in a box under an uniform acceleration is studied. It is found that the horizontal profile evolves to a final equilibrium profile which corresponds to a tilted straight plane whose slope is a function of the magnitude of the imposed acceleration, the magnitude of the gravity acceleration and the friction coefficient of the granular material. Here is presented a simple model, based on the Coulomb's law, which describes correctly such a deformation and some experiments that back the theoretical predictions.

Keywords: Static sandpiles; avalanches; pattern formation.

En este trabajo se estudia el problema de la deformación de la superficie libre de un material granular no cohesivo confinado en una caja bajo una aceleración uniforme. Se encuentra que el perfil horizontal evoluciona a un perfil de equilibrio que corresponde a un plano inclinado cuya pendiente es una función de la magnitud de la aceleración impuesta, la magnitud de la aceleración de la gravedad y el coeficiente de fricción del material granular. Se presenta un modelo simple, basado en la ley de Coulomb, el cual describe correctamente tal deformación y algunos experimentos que respaldan las predicciones teóricas.

Descriptores: Montones estáticos de arena; avalanchas; formación de patrones.

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1. Introduction

In his book on *Hydrodynamics* Daniel Bernoulli [1] analyzed, near three centuries ago, the problem of the free surface deformation of the initially horizontal level of quiescent water in a cylindrical vessel when it is accelerated uniformly along the horizontal direction. He found that the profile changed to a tilted straight plane whose slope is a function of the gravity acceleration, g, and the acceleration imposed to the vessel during a rigid body translation, a^*. Today, the comprehension of this problem is very important to design safe structures of vehicles carrying liquid cargo and to analyze their driving stability when they are subjected to episodes of sudden breaking or lane change maneuvers [2-5].

A set of similar problems to the previously alluded occur in vehicles carrying dry cohesionless granular materials [6] but some important differences appear due to the role played by friction. Thus, the motivation of this study is the determination of how the free surface of a non cohesive granular material in a box changes its shape to final states of equilibrium due to the one-dimensional, horizontal uniform acceleration.

The modeling of this problem can be made through the formulation of a balance of force equation which uses the Coulomb's law; it gives a relation between the tangential (F) and normal (N) forces on a unit volume at the free surface of a non cohesive granular material in sliding contact [7]. This formula establishes that the equilibrium condition (the slope) is maintained if

$$F \leq \mu N. \tag{1}$$

The motion occurs if and only if there is equality, the quantity μ is the coefficient of friction of the material. In general terms the start of the distortion of the free surface occurs when the slope of the heap reaches a maximum value, the angle of internal friction φ_c. In this paper the sliding contact condition may be reached under the imposed acceleration, a^*, and the resulting motion equation in this case gives the final state of deformation which, as in the case of accelerated liquids, corresponds to straight slopes if the initial free surface is horizontal and the magnitude of the normalized acceleration $a = a^*/g$ overcomes the friction coefficient $\mu = \tan\varphi_c$.

In this work the problem of the change of the slope of the free surface is studied theoretically and experimentally. The theoretical treatment presented in the next section allows giving analytical expressions for the resulting slope, the mobilized volume due to this deformation and the new position of the center of mass. Moreover, the model lets to analyze qualitatively much more complex situations as the distortion of non horizontal free surfaces. In Sec. 3 it is discussed experimentally the change of the slope of mustard seeds under different initial conditions and quantitative and qualitative comparisons back the predictions of the analytical model. Finally, in Sec. 4 the main conclusions and a discussion on the limitations and perspectives of this study will be given.

2. Theory

Here it is supposed that a cohesionless granular material partially fills a box, $2L^*$ length and w^* width, whereas the height of the granular material, respect to the basis (y^*=0), is H^*.

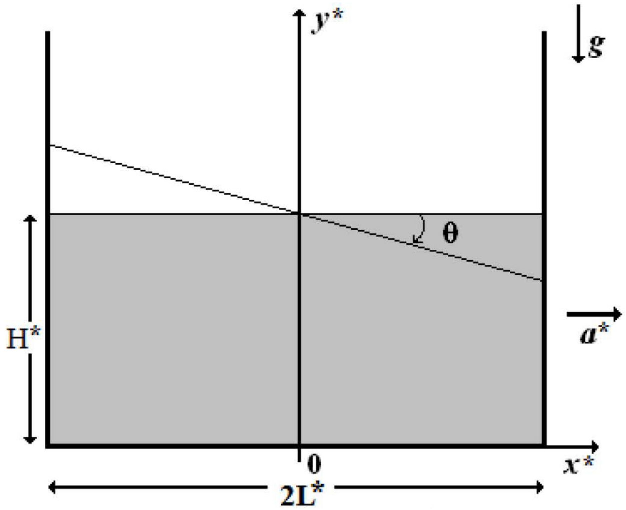

FIGURE 1. Schematic of the accelerated box filled with a cohesionless granular material. The initial profile is a horizontal plane and when the box is subjected to the acceleration, a^*, the equilibrium profile is an inclined plane at the angle θ.

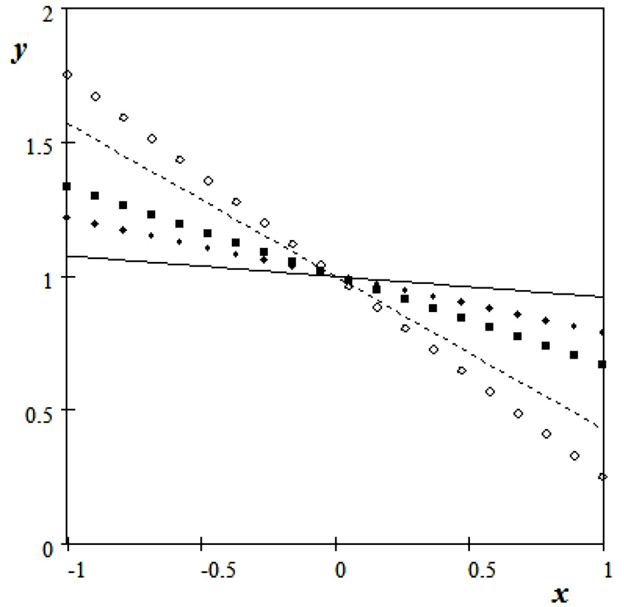

FIGURE 2. Dimensionless two-dimensional profiles of the free surface under uniform accelerations: $a = 0.6$ (continuous line), $a = 0.8$ (symbol □), $a = 1$ (symbol ■), $a = 1.5$ (dashed line) and $a = 2$ (symbol ◇).

See Fig. 1. The analysis of the deformation of the free surface considers the presence of the gravity acceleration and a one-dimensional, horizontal, uniform acceleration on the box that provokes the initial horizontal profile changes up to a tilted straight plane that does not is deformed anymore (the equilibrium profile has been attained); first the slope of this plane is computed and afterwards the mobilized volume of material and the center of mass are also estimated for different values of the acceleration.

2.1. The free surface deformation in a box on the floor

In Fig. 1 a scheme of the problem is depicted. Initially, the granular material has a constant level $y^* = H^*$ in the box and suddenly it is uniformly accelerated to the right hand side with an acceleration a^*, respect to the inertial system fixed to the floor. Since the point of view of an observer in the (x^*, y^*, z^*) system, fixed to the box, there exits an acceleration a^* in the opposite direction which deforms the free surface as is sketched in Fig. 1. It is well accepted that when this type of surface deformation occurs it is possible to build the motion equation using Eq. (1) [8-10]. Thus, for an element of volume of bulk density ρ, at the free surface making an angle θ respect to the horizontal (Fig. 1), the motion equation is

$$\rho\left(a^* \cos\theta - g \sin\theta\right) = \mu\rho\left(a^* \sin\theta + g\cos\theta\right). \quad (2)$$

Rearranging terms, scaling the coordinates with the half-length of the box L^*, $i.e.$, (x, y, z)= $(x^*/L, y^*/L, z^*/L)$, and introducing the dimensionless acceleration $a = a^*/g$, we obtain the dimensionless differential equation for the slope

$$\frac{dy}{dx} = \frac{\mu - a}{1 + \mu a}, \quad (3)$$

which is valid for any plane $z = const.$, where $0 \leq z \leq w$ and $w = w^*/L^*$. The solution of Eq. (3), yields the profile

$$y(x) = \frac{\mu - a}{1 + \mu a}x + H, \quad (4)$$

where $H = H^*/L$ and to determine the constant resulting from the solution of the differential equation has been used the overall mass conservation of granular material. Equation (4) allows to notice that there is a critical value of the dimensionless acceleration, $a = \mu$, below of which the surface is no deformed. If the acceleration overcomes this threshold value then the initially horizontal profile will be deformed to a surface whose profile is a straight plane, with slope ($\mu - a$)/(1+μa), which always crosses the point $y = H$, the initial height, at the center of the box.

In Fig. 2 are plotted some nondimensional equilibrium profiles assuming that the granular material has a friction coefficient $\mu = 0.5$ and that it is contained in a box with dimensions: length equal to 2 and width w, and initial level of filling $H = 1$. The dimensionless accelerations were assumed as $a = 0.6$, 0.8, 1, 1.5 and 2. In such a figure it is clear that the inclination of the resulting profiles increases if a does it. Note that as in a liquid, if $\mu = 0$, the surface profile given by Eq. (4) transforms in

$$y(x) = -ax + H, \quad (5)$$

which always yields a deformation of the free surface for any value of a, $i.e.$, there is no a critical value of acceleration as in the case a granular material.

Due to in a non cohesive granular material the number of initial configurations is practically infinite, whenever the local slope has an angle $\theta < \varphi_c$, which does not occurs with

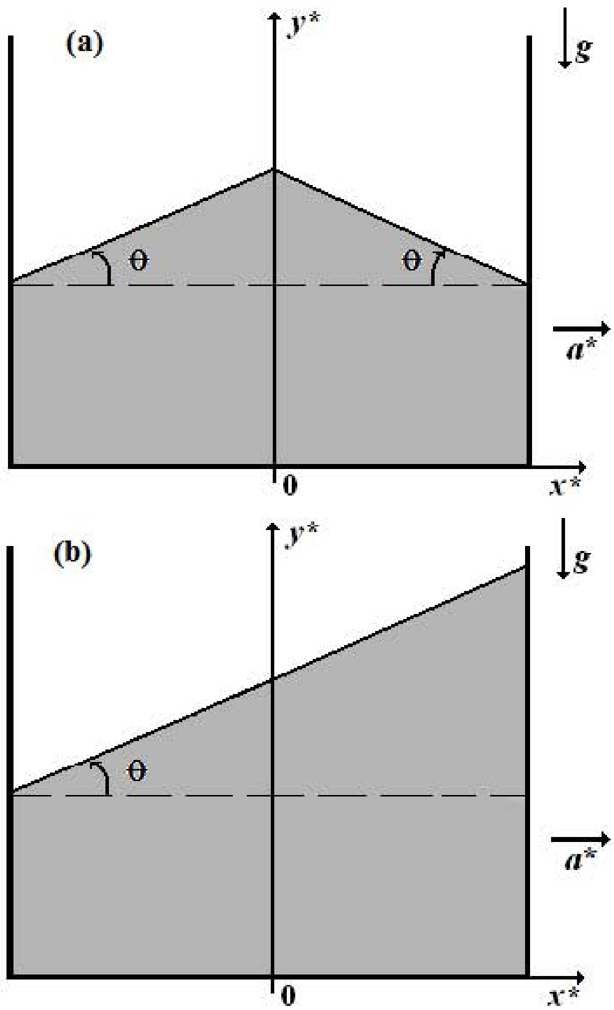

FIGURE 3. Schematic of other initial conditions: (a) symmetrical heap and (b) asymmetrical heap. During acceleration the regions with positive slope are immediately deformed.

liquids, it is important to discuss two cases of interest that allows to notice fundamental aspects of the free surfaces of accelerated systems. First.- when the initial configuration of the free surface is a symmetric heap, as the one shown in Fig. 3(a), the region where $-L^* \leq x^* \leq 0$ has a positive slope which during accelerated motion also obeys the Eq. (3), but in a different sense.

It means that if $a < \mu$, does not matter how small is a, the free surface in that region will change to a lower slope. This motion necessarily will change the profile in the zone $0 \leq x^* \leq L^*$ and thus the complete free surface will be deformed in a complex way. The study of such a problem is far from the reach of this work because it involves the coupled motion of both tilted free surfaces. More specifically, this problem involves rearrangements deep inside the bulk and its treatment requires extensive numerical computations for the bulk such as the discrete element method [6]. Second.- a similar situation occurs when the free surface is a tilted plane making an angle θ respect the horizontal, as in Fig. 3(b). When the box is accelerated to the right hand side the granular material in the free surface felt a normal force that tends to decompress the heap and a downhill granular flow occurs on its free surface. Consequently a change in the free surface profile will occur to any value of a. Both of these configurations allow concluding that knowledge of the initial profile is essential to know the changes of the complete free surface itself, it is also other different situation respect to the classical case of the free surface deformation of a liquid [7]. In next section the mobilized volume and the new position of the center of mass due to the free surface deformation are estimated.

2.2. The mobilized volume and the motion of the center of mass

The granular material in a box changes its slope from the horizontal to a tilted slope if $a > \mu$ in accordance with Eq. (3) and, in a first approximation, it is assumed that in such a case the bulk density is the same as in the initial stage [8-10], thus the volume of mobilized material from the right hand side to the left hand side, see Fig. 1, has the dimensionless form

$$V_m = w\left[\frac{a-\mu}{2(1+\mu a)}\right], \quad \text{if} \quad a > \mu, \tag{6}$$

and the fraction of mobilized volume relative to the total dimensionless volume V_T*=2wH*, is given by

$$V_{fm} = \frac{V_m}{V_T} = \frac{a-\mu}{4H\,(1+\mu a)}, \text{ if } a > \mu. \tag{7}$$

Clearly, the mobilization of granular material changes the original position of the center of mass of the granular material in the box, moreover, this quantity can be of interest to estimate the net torques on tires in vehicles transporting grains [6]. Assuming, once again, that the density does not changes under the deformation of the free surface, it is direct to found that the center of mass can be computed from the relation

$$\mathbf{r}_{cm} = \frac{\int \mathbf{r}dxdydz}{V_T} \tag{8}$$

where $\mathbf{r}$=(x, y, z). Using Eq. (8) and Eq. (3) to determine the boundaries that limits the localization of the granular material, is obtained that

$$\begin{aligned} x_{cm} &= \frac{w\int\limits_{-1}^{1}\int\limits_{0}^{y(x)} xdydx}{2Hw} \\ &= \frac{\mu-a}{3\,(a\mu+1)\,H}, \quad \text{if} \quad a > \mu, \end{aligned} \tag{9}$$

$$\begin{aligned} y_{cm} &= \frac{w\int\limits_{-1}^{1}\int\limits_{0}^{y(x)} ydydx}{2Hw} \\ &= \frac{3H^2\,(a\mu+1)^2 + (a-\mu)^2}{6H\,(a\mu+1)^2}, \quad \text{if} \quad a > \mu \end{aligned} \tag{10}$$

$$z_{cm} = \frac{w}{2}. \tag{11}$$

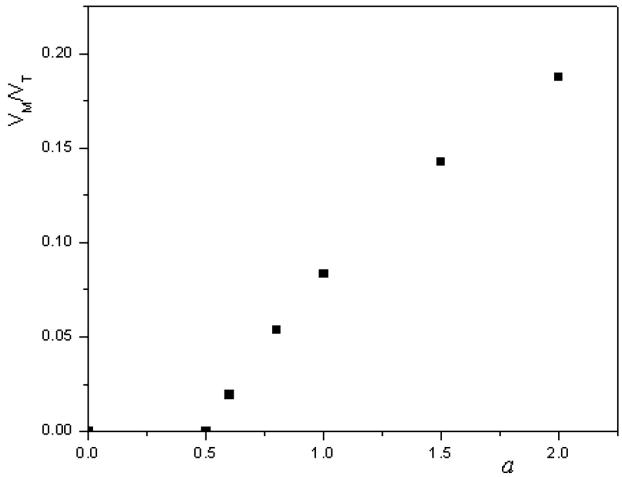

FIGURE 4. Fractional volumes as a function of the dimensionless acceleration, a. In this plot have been considered the values of a given in Fig. 2.

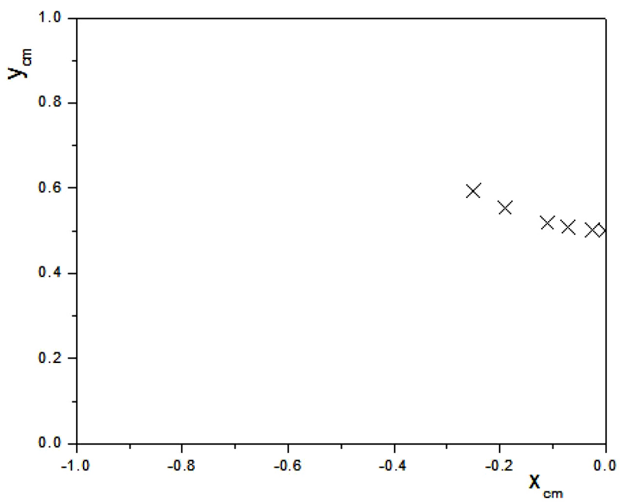

FIGURE 5. Position of the center of mass for the heaps deformed under the dimensionless accelerations given in Fig. 2. The point in $x = 0$ corresponds to a non deformed heap with a horizontal profile.

It is useful to discuss the behavior of V_{fm} and $\mathbf{r}_{cm}$ in a general context. In Fig. 4 are plotted the fractional volumes, V_{fm}, assuming the same values of the dimensionless acceleration used in the previous section.

From Fig. 4 is easily estimated that the fractional mobilized volume when $a = 2$ is $V_{fm} = 0.1875$, $i.e.$, 18.75% but for the case of a liquid (μ=0 (in Eq. (6)) this quantity achieves 50%. This latter result illustrates the important role played by friction which limits the motion of the bulk and eventually allows different initial configurations. In Fig. 5 the position of the center of mass, in the plane *xy* and at $z = w/2$ is plotted under the same conditions: each point indicates the localization of the center of mass when the equilibrium profile for a given acceleration has been achieved.

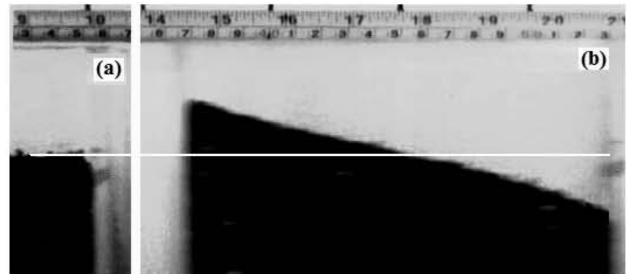

FIGURE 6. a) Partial view of a heap of mustard seeds confined in a rectangular box with a profile initially flat and (b) equilibrium profile of the heap after the acceleration, of magnitude a =1.22, has been imposed. The horizontal white line indicates the position of the initial level of the heap.

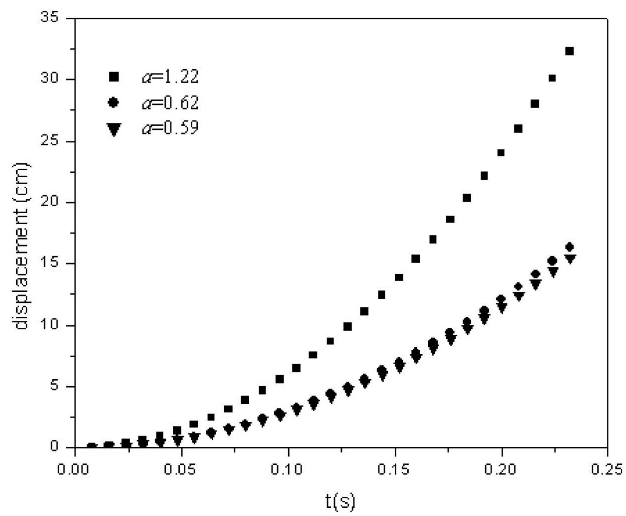

FIGURE 7. Plot of the accelerated motion of boxes containing mustard seeds: for a=1.22 (symbol ■) the initial profile was horizontal and it corresponds to the deformation shown in Fig. 4. For $a = 0.62$ (symbol ●) there is no deformation of the free surface because the initial profile was horizontal and for $a = 0.59$ (◇) the heap in the box is initially symmetric and despite $a < \mu$ in this case occurs an overall deformation.

3. Experiments

It is direct to carry out experiments to validate mainly the theoretical predictions for the surface profiles. In experiments were used mustard seeds which were confined in an acrylic-walled rectangular box, 16.3 cm length and 5 cm width. Mustard seeds have nearly spherical shape and thus their average diameter was d_g =1.5±0.1 mm, density $\rho_g = 1.3$ gr/cm^3, the friction coefficient was μ=0.62±0.01, and the angle of internal friction took the value $\varphi_c = 32°$±0.5° [11].

Due to the low density of the mustard seeds experiments of one-dimensional, uniform accelerations of different magnitudes were made in the box mounted on a horizontal, linear air track. In each experiment the box containing the mustard seeds was pulled, almost frictionless, by a mass, located at the opposite edge, and falling down freely. The motion of the

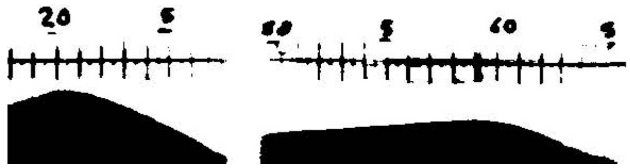

FIGURE 8. The picture on the left hand shows partially the symmetric heap to appreciate the deformation of an initially symmetric heap of mustard seeds when the dimensionless acceleration is $a = 0.59$, *i.e.*, this value is lower than that of the friction coefficient, $\mu = 0.62$. It shows that if the initial condition has a zone with a positive slope a free surface deformation can be achieved.

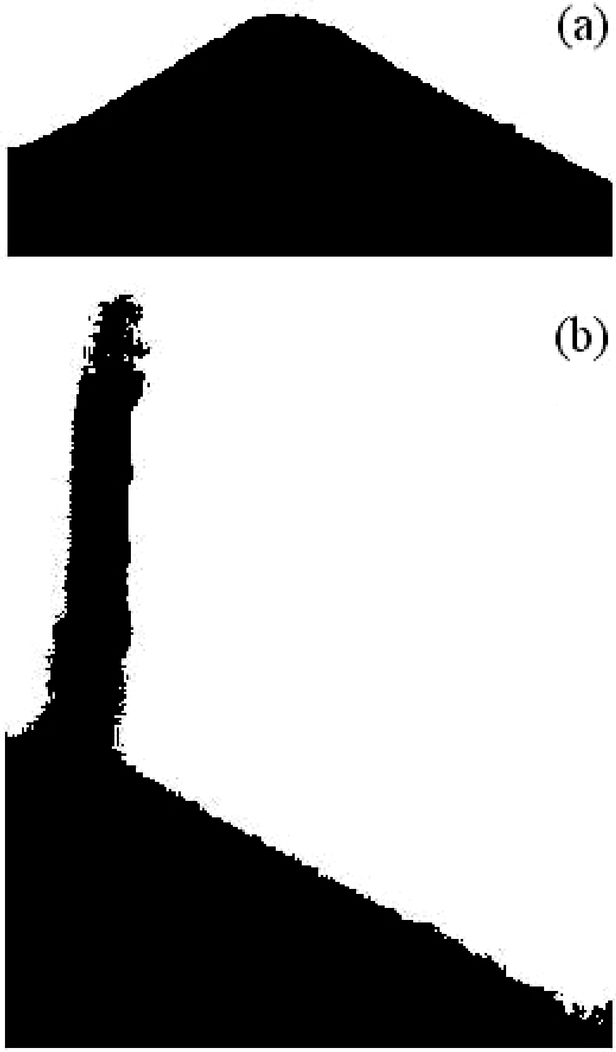

FIGURE 9. (a) Partial view of the symmetric heap and (b) image of a granular jet occurring at $a \sim 0.7$

free surface during acceleration events was video-recorded with a high speed camera model Redlake HG-100K. All video recordings were taken to 125 fps to follow sequentially the free surface deformation and the motion of the box.

In a first experiment the box was filled up to $H = 0.48$ cm with mustard seeds and on it was imposed a dimensionless acceleration equal to $a = 1.22$, clearly the horizontal free surface will be deformed because $a > \mu$. The determination of the acceleration was made using the formula $s = (at^2/2)$ where s is the displacement of a point of the box at a time t. In the experiment the average value of y at each edge of the box was $\overline{y} \approx 0.245$ (see Fig. 6) meanwhile using formula (3) it is found that $y = 0.247$,$i.e.$, a relative error of around 1%. In Fig. 7 is shown the plot of the displacement as a function of time, t, to quantify the acceleration of the box. In such a plot there are other values of α which correspond to a case where the free surface is horizontal and despite acceleration is $a = 0.62$ there is not observed a macroscopic free surface deformation. Finally the case where $a = 0.59$ corresponds to the acceleration of a box with a symmetric heap, as is shown in Fig. 8. There is appreciated that a substantial deformation occurs, even that $a < \mu$. It confirms that during a small acceleration the zone with positive slope can be mobilized because there the normal force plays a decompressing role, instead the compressive one occurring in the zone with negative slope.

For a more high value of acceleration can be observed a jet as the one shown in Fig. 9 where is observed that the initial condition was also of a symmetric heap, such a jet have appeared when $a = 1.7$. The jet was generated due to the strong uphill surface flow, which climbs the wall of the box, and as the box is decelerated the jet moves forward and eventually disappears.

4. Conclusions

In this work has been analyzed the problem of the free surface deformation of a non cohesive granular material confined in a box under a uniform, one-dimensional acceleration. Using a simple model based on the Coulomb' s law of friction it has been shown that for an initially horizontal profile there exists a critical value of the dimensionless acceleration, $a = \mu$, above of which the acceleration allows to achieve a final, equilibrium profile corresponding to a tilted straight plane. However, if the initial profile has a zone with a positive slope, this condition does not is maintained and it is possible to get a deformation that involves rearrangements deep inside granular material. In this latter case, theory here reported only predicts qualitatively this fact. Experiments using mustard seeds back all these results and also show complex evolutions of the bulk material, as the presence of jets when acceleration is very strong.

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