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Mass flow rate of granular material flowing from tilted bins

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ABSTRACT

We report experiments performed to describe the behavior of the experimental mass flow rate of cohesionless granular material, $m_{\beta\mathrm{expt}}$, through circular orifices of diameter D made on sidewalls of tilted bins. In such experiments, the influence of the wall thickness of the bin, w, and the tilt angle with respect to the vertical, β, were also regarded. The experimental measurements, using beach sand and granulated sugar, yield a linear correlation among $m_{\beta\mathrm{expt}}$ and a theoretical piecewise correlation of the mass flow rate, m_β which is valid for the overall range of values of β.

1. Introduction

Silo discharge through a bottom circular outlet is one of the oldest and most widely studied problems in granular flow owing to the simple setup and geometry of the system [1,2,3,4,5,6,7,8,9,10]. It has been extensively investigated both experimentally and computationally [11,12,13,14], and many granular, gravity flow theories use silo discharge as a benchmark for validation.

In experiments, modern visualization techniques, mainly those based on digital particle image velocimetry (DPIV) have shown that the inner flow of non-cohesive granular materials during the discharge of silos from bottom symmetric and asymmetric outlets is very complex [15,16,17,18,19,20,21,14,22], and the mass flow rate typically fluctuates around its mean value, $\dot{m}$. Experimental studies of this type have been performed in plane models of silos, where plane flows are developed. Despite this limitation, through DPIV some works [14,18,19,21,22] allow showing that the measured velocity fields correlate very well with the phenomenological formulas for the mass flow rate, which essentially follows the Beverloo formula for rectangular exit holes [3].

By the way, the study of the discharge rate from orifices on vertical or tilted sidewalls of silos has been largely neglected perhaps due to the asymmetric flow profile which occurs close to the sidewalls. The tilting of the sidewalls deepens such asymmetry, and can intensify the flow rate of grains through the increase of the effective area of outflow and the gravity action. This is the reason why we need to tilt the sugar storage container to pour just the right amount of such substance. Thus, the main goal of this paper is to understand phenomenologically the leading factors that regulate the continuous dosification of grains through the inclination of the bin. The discontinuous dosification and clogging of the granular material, by using tilted bins, is related, in first instance, to the use of small orifice diameters and large inclinations and it has been studied elsewhere [23,24,25].

As a starting point, it is convenient to notice that in recent works [26,27] we derived a formula valid to estimate the mass flow rate through orifices on the sidewalls of bins, $m(D,w)$, that regards the dependence on the orifice diameter D and the wall thickness w for granular solids in the limit $D \gg d_g$, *i.e.*, from a continuum description where the role of the mean grain diameter, d_g, is not considered. In such a case, we found that the mass flow rate scales as $\dot{m}(D, w) \propto \rho g^{1/2} D^{5/2}(\alpha - \theta_r)$, where ρ is the bulk density of the granular material, θ_r is its angle of repose, g is the acceleration due to gravity and α is the angle that characterizes the orifice, which we defined as $\alpha = \arctan(D/w)$. Such a formula is strictly valid for the discharge rates of granular solids through vertical walls of any wall thickness. It also predicts the critical thickness, for a given diameter D, for which the granular flow will be arrested and is valid even for the ideal case where the wall thickness vanishes, namely, walls without thickness. This latest case is very interesting because, in a laboratory, many experiments of gravity flow use bins and silos made

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with very thin walls. Also, in numerical simulations, a null wall thickness consideration is a typical case.

Finally, it is important to notice that for the case of zero wall thickness and any diameter, the angle α reaches the value $\alpha = \arctan(D/0) \to \pi/2$, and consequently $\dot{m}(D, w = 0) \propto \rho g^{1/2} D^{5/2}[(\pi/2) - \theta_r)]$, which means that in this case, the essential dependence of $\dot{m}$ on D yields the classical term $D^{5/2}$, as has been reported in some experiments with lateral orifices [2,28,23]. For the case in which the wall thickness is larger than the diameter, $(D/w) < 1$, the mass flow rate $\dot{m}(D, w) \propto \rho g^{1/2} D^{5/2}(\arctan(D/w) - \theta_r)$ yields the approximate formula $\dot{m}(D, w) \propto \rho g^{1/2} D^{7/2}/w$ which was reported on the earliest experimental works on lateral orifices [29,30,31].

All the above results have been detailed in order to show a clear difference among the formulas for the mass flow rate of horizontal circular orifices of diameter D, $\dot{m}_0(D)$, which essentially is given by the correlation $\dot{m}_0(D) \propto \rho g^{1/2} D^{5/2}$ and that for holes on vertical sidewalls.

In this work we report a series of experiments on the discharge for tilted bins in order to quantify the dependence of the mass flow rate on β, the tilt angle with respect to the vertical. In the present study, bins having circular orifices on the side faces were inclined, at specific angles, clockwise from the vertical to the horizontal position (positive angles) and, counterclockwise, from the vertical up to the tilted position where the granular flow is arrested (negative angles). To our knowledge, studies of this type are very scarce: in fact, Franklin and Johanson [2] studied the effect of the inclination on the flow rate by taking into account the horizontal and vertical cases and another two intermediate angles of inclination. In such a case the wall thickness of the plate having the holes was $w = 2.38$ mm. As a result, they proposed a relation for the mass flow rate, as a function of the tilt angle θ (measured to the horizontal), where the wall thickness was not taken into account (a more detailed discussion will be given in next section). Later on, Sheldon and Durian [23] also reported measurements of flow rates and clogging from tilted containers where the Franklin and Johanson's correlation remained as valid. Again, perhaps it was due to their wall thinness ($w = 0.25$ and $w = 3$ mm, respectively).

Here, we will consider the fact that the bin sidewalls have a finite thickness, and that will consequently bias the determination of the mass flow rate at a given tilted angle, β. In summary, we will contrast through experiments our new findings for the flow rate in tilted bins and those published by other authors [23,32]. A general correlation valid for the whole range of the inclination angles where flow occurs will be featured.

To reach our goals, the plan of this work is as follows. Firstly, in the next section we bring up a correlation apparently valid for experiments of discharge rates from holes in tilted bins after Franklin and Johanson [2]. Afterwards, in Section 3, we will describe a series of experiments performed with tilted bins. Then, in Section 4, we propose and discuss, on the basis of the experimental results, a correlation that describes correctly a wide range of angles and embraces the changes in D, w and β. Finally, in Section 5, we give the main conclusions of the study tackled here.

2. The Franklin and Johanson correlation

Sixty years ago, Franklin and Johanson [2] established, by using logical reasoning, that the mass flow rate from a tilted side wall with circular orifices of diameter D, at the angle θ with the horizontal, is given by the relation

$$\dot{m}_\theta(D, \theta) \propto \dot{m}_0(D)\left(\frac{\cos\theta_r + \cos\theta}{\cos\theta_r + 1}\right). \tag{1}$$

In correlation (1) the angle θ is measured counterclockwise and $\dot{m}_0(D)$ is, as mentioned in the Introduction, the mass flow rate through horizontal orifices of diameters D, which is essentially given by Hagen's law [1]

$$\dot{m}_0(D) \propto \rho g^{1/2} D^{5/2}. \tag{2}$$

It is important to comment that the relation among the experimental measurement of the mass flow rate, $\dot{m}_{0\text{expt}}$, and the mass flow rate $\dot{m}_0(D)$ for horizontal orifices given above, yields a linear relation of the form [3,28],

$$\dot{m}_{0\text{expt}} = a\dot{m}_0(D) = a\rho g^{1/2} D^{5/2}, \tag{3}$$

where a is the dimensionless discharge coefficient.

Here it must be noted that the symbol "$\propto$" establishes the proportionality between the left-hand side and the right-hand side terms of a given relation, meanwhile the symbol " $=$ " states a linear trend between the measured and predicted flow rates. Similar considerations have been used elsewhere [7,28], among others. Finally, we will use this notation for the rest of the paper, unless otherwise specified.

In order to reach further understanding of relation (1), it is important to examine some significant limits. First: clearly if $\theta = 0$,

$$\dot{m}_{\theta=0} \propto \dot{m}_0(D), \tag{4}$$

i.e., the case with horizontal orifices is recovered. Second: if $\theta = \pi - \theta_r$ then

$$\dot{m}_{\theta=\pi-\theta_r} = 0, \tag{5}$$

and the granular flow has an abrupt halt. Third: for the case $\theta = \pi/2$, relation (1) yields that the flow rate for the vertical case is directly obtained from the flow rate in the horizontal case by simply using the multiplying factor $\cos\theta_r/(\cos\theta_r + 1)$, *i.e.*,

$$\dot{m}_{\theta=\frac{\pi}{2}} \propto \dot{m}_0(D)\left(\frac{\cos\theta_r}{\cos\theta_r + 1}\right). \tag{6}$$

By the way, as was discussed in the Introduction, the valid correlation for the mass flow rate of granular material emerging from lateral, vertical side walls of silos, with zero wall thickness (where $\alpha = \pi/2$) and any diameter D, is given by [26,27]

$$\dot{m}\,(D, w = 0) \propto \dot{m}_0(D)\left[\frac{\pi}{2} - \theta_r\right]. \tag{7}$$

Clearly, correlations (6) and (7) are not in any sense, equivalent, due to their dependence on θ_r. Despite this, some of these formulas should predict correctly the mass flow rate from orifices on vertical side walls. In order to discern the correctness of those formulas we must insist that correlation (6) was obtained through logical reasoning while in the Introduction, we have soundly established correlation (7) as a formally obtained limit case from a more general formula where no necessarily the wall thickness vanishes [26,27]. We have shown that the correct correlation for the flow rate of vertical orifices must include information about the wall thickness [26,27], *i.e.*,

$$\dot{m}\,(D, w) \propto \dot{m}_0(D)\left[\arctan\left(\frac{D}{w}\right) - \theta_r\right], \tag{8}$$

where $\dot{m}_0(D)$ is given by formula (2).

Finally, the correlation among the experimentally measured mass flow rates for vertical orifices, $\dot{m}_{\text{expt}}(D, w)$, as a function of the wall thickness and the diameter, is given by [26,27]

$$\dot{m}_{\text{expt}}(D, w) = c\,\dot{m}\,(D, w) = c\dot{m}_0(D)\left[\arctan\left(\frac{D}{w}\right) - \theta_r\right], \tag{9}$$

where $\dot{m}(D,w)$ is given by correlation (8) and now c is the experimental dimensionless discharge coefficient for lateral orifices.

To achieve a reliable formula, equivalent to Eqs. (3) and (9), but for tilted bins and valid for a wide range of tilt angles, we have performed a set of experiments that involves changes in D, w, θ_r and also in β. In the next section we describe such experiments.

3. Experiments

In the current work, the simultaneous effects of D, w, β and θ_r on the mass flow rate are examined. As in our previous studies [26,27], here, we also used well characterized dry noncohesive materials like beach sand (composed of irregular grains of mean diameter, bulk density $\rho = 1.5 \pm 0.01$ g/cm^3 and angle of repose $\theta_r = 33° \pm 0.5° = 0.57 \pm 0.008$ rad) and, granulated sugar (mean diameter $d_g = 0.073$ cm, bulk density $\rho = 0.84 \pm 0.01$ g/cm^3 and angle of repose $\theta_r = 33.5° \pm 0.5° = 0.58 \pm 0.008$ rad). Those materials outflow from staggered circular orifices of diameters $D = 0.90 \pm 0.05$, 1.00 ± 0.5 and 2.00 ± 0.05 cm, respectively, made on the side faces of a bin having walls $w = 0.3$, 0.4, 0.6 and 0.9 cm nominal thicknesses. In our experiments we found that the mean wall friction coefficient (measured with the tilting plate method [6]) among granulated sugar and acrylic was $\mu_w = 0.70 \pm 0.01$ and among sand beach and acrylic it was $\mu_w = 0.80 \pm 0.01$. Finally, experiments were performed within a climate-controlled laboratory (25 ± 1 °C and 45 ± 10% R.H.).

The transparent acrylic-made bin had 10×10 cm^2 inner cross-section and 50 cm height. A schematic of the experimental setup is given in Fig. 1 where the bin is tilted up to the desired angle β to the vertical. Positive angles are measured clockwise. To measure the mass flow rate a digital force sensor model Pasco CI-6537 (with a resolution of 0.03 N) was used. For positive angles the reservoir attached to the sensor is located in front of the orifice. For not so large negative angles the granular material enables the existence of a flow and, in such cases the reservoir attached to the sensor is located just at the lower rim of the exterior sidewall. Pictures of the granular flows corresponding to these latest cases are shown in Fig. 2, where the angle β, w and D are also shown for several orifices.

Details of the measurement procedure of the discharge rates by using the force sensor are given elsewhere [26,27] but, essentially, the sensor measures the instantaneous weight, $W(t) = m(t)g$, where $m(t)$ is the instantaneous mass of the bulk material. The time derivative of weight provides that $dW/dt = g(dm/dt)$, hence the mass flow rate is computed as $\dot{m}_{\text{expt}} = (dW/dt)/g$. In each experiment the vertical bin was filled by pouring the grains within it and then the bin was tilted to the desired angle β, which was measured with a digital clinometer whose minimum scale is 0.5° (0.008 rad). Measurements of the mass flow rates were carried out ten times for each value of β. Finally, each plotted value of $\dot{m}_{\beta\text{expt}}$ corresponds to an average of the ten measurements.

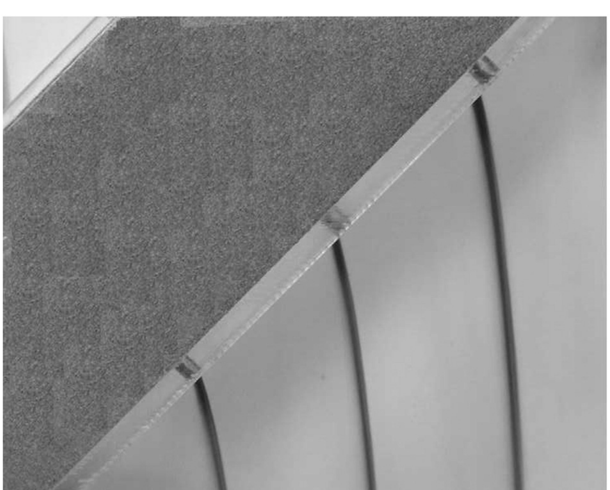

Fig. 1. Scheme of the experimental setup. The circular orifices are perpendicular to the sidewalls.

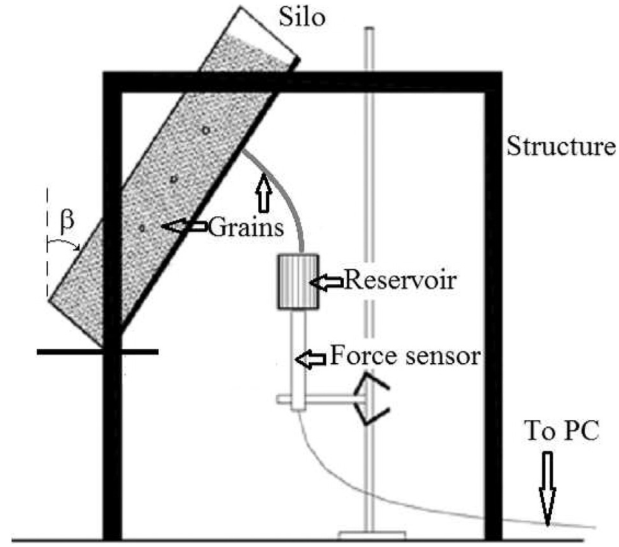

Fig. 2. Snapshots of sand emerging from orifices on sidewalls in tilted bins: (a) simultaneous sand jets from staggered orifices for a tilt angle $\beta > 0$. (b) Granular flow from a bin with a tilt angle $\beta < 0$.

In summary, here we show and discuss results of the mass flow rate measurements for very dissimilar cases: in one case we used beach sand in bins with sidewalls $w = 0.3$, 0.6 cm thickness and orifice $D = 1.00 \pm 0.05$ cm. In the other one we used granulated sugar with sidewalls $w = 0.3$, 0.9 cm thickness and orifice $D = 2.00 \pm 0.05$ cm. We chose such diameters because granulated sugar has a mean diameter 2.4 times the mean diameter in experiments with sand grains. In Fig. 3 we plot the mass flow rate for several tilt angles β. In Fig. 3(a) we show the plots for beach sand and in Fig. 3(b) the plots correspond to granulated sugar. In both plots we observe several physically meaningful facts: firstly the mass flow rates behave as sine-like functions of β and (for a fixed diameter) at small tilt angles the mass flow rate is stronger for the smaller wall thicknesses, which appears logical but, conversely, at large tilt angles the mass flow rates are slightly larger or even near equal to the thicker wall. Secondly, in both cases, the flow rates are stronger as $\beta \to \pi/2$ and, at $\beta = \pi/2$, the value of $\dot{m}_{\beta\text{expt}}$ for different thicknesses and equal orifice diameters is the same, which confirms that for horizontal orifices the wall thickness does not influence such quantity. Thirdly, in all cases the flows of sand and sugar were all arrested at different negative *critical inclination angles*, β^*, which depends on the granular material used (its angle of repose) and the wall angle α. In experiments, the measurement of the angles of arrest was made by tilting gently, counterclockwise, the bin having opened the orifice of interest and when the flow was arrested we measured the corresponding negative angle β^*. This procedure was repeated ten times for each orifice. Here, we report the mean values of β^*: for sand outflowing from orifices $D = 1.00 \pm 0.05$ cm diameter, $\beta^* \approx -0.641 \pm 0.008$ if $w = 0.3$ cm and $\beta^* \approx -0.391 \pm 0.0.008$ if $w = 0.6$ cm. For granulated sugar emerging from exit holes $D = 2.00 \pm 0.05$ cm diameter, $\beta^* \approx -0.773 \pm 0.008$ if $w = 0.3$ cm and $\beta^* \approx -0.499 \pm 0.008$ if $w = 0.9$ cm.

In the next section we will use our results to get a reliable formula to estimate the mass flow rate and other properties of the granular flow from tilted bins.

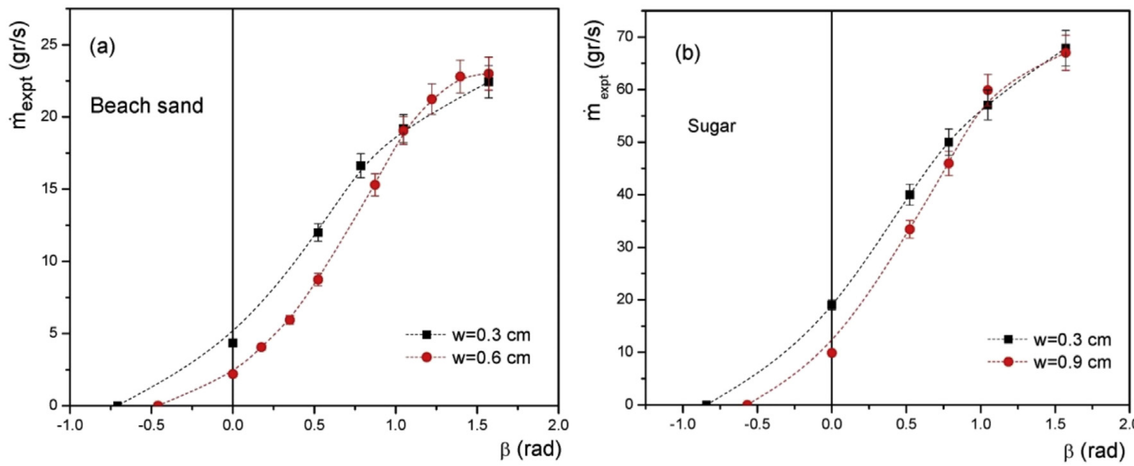

Fig. 3. Plots of the measured mass flow rates in tilted bins, $\dot{m}_{\beta\ \mathrm{expt}}$, as a function of the tilt angle β. In (a) we used beach sand, with orifices $D = 1.00 \pm 0.05$ cm in diameter and sidewalls with thicknesses of $w = 0.3$ cm and $w = 0.6$ cm, respectively. In (b) granulated sugar was used and in this case we report results for holes $D = 2.00 \pm 0.05$ cm in diameter and sidewalls with thicknesses of $w = 0.6$ cm and $w = 0.9$ cm. For this case the data seems to obey a sine-like behavior.

4. A general correlation

The search of a general correlation containing the dependence of the flow rate $\dot{m}_\beta$ on D, w and β deserves a careful analysis. At first glance, it is easy to conclude that if the vertical wall is very thick, there will be no efflux of granular material through the orifices. Thus, the influence of the wall thickness, w, must be present every time the granular flow occurs. Here, we introduce a correlation that is valid for a wide range of values of β, including the cases $\beta = 0$ (orifices on vertical sidewalls) and $\beta = \beta^*$ (the critical negative inclination when the granular flow is arrested). Moreover, it must be a sine-like function if we are to take into account the experimental behavior given in the previous section (see Fig. 3). Therefore, we propose that

$$\dot{m}_\beta(D, w, \beta) \propto \dot{m}_0(D)[\beta + \alpha - \theta_r] \frac{\sin(\beta + \alpha - \theta_r)}{\sin(\alpha - \theta_r)}, \qquad (10)$$

where $\dot{m}_0(D)$ is the Hagen's law given by relation (2).

Correlation (10) is an extension of relation (8). In relation (10) we have added the angle of inclination β to the term $\alpha - \theta_r$, because the positive inclination enhances the outflow. Conversely a negative inclination weakens it. The normalized sine-like function $\sin(\beta + \alpha - \theta_r)/\sin(\alpha - \theta_r)$ in relation (10) is introduced to recover the valid formula for the vertical cases.

Formula (10) immediately yields, as a particular case, the condition for which the flow is completely arrested. It is the equation

$$\dot{m}_{\beta^*} = 0, \quad \text{when} \quad \beta^* = -(\alpha - \theta_r), \qquad (11)$$

which was obtained if $x = (\beta + \alpha - \theta_r) \to 0$ in relation (10) [33], and then $\beta^* = -(\alpha - \theta_r)$. Incidentally, when the wall thickness is zero, the wall angle is $\alpha = \pi/2$, and thus from Eq. (11), $\beta^* = -(\pi/2 - \theta_r)$. As aforementioned, in the Franklin and Johanson approach [2] the arrest condition $\dot{m}_\theta = 0$ occurs if $\theta = \pi - \theta_r$. For these latest cases the angles β and θ are related through the transformation $\beta = (\pi/2) - \theta$, so it is direct to check that both arrest conditions are the same.

For the case of vertical bins, if we take $\beta = 0$ in correlation (10), it allows getting

$$\dot{m}_{\beta=0} \propto \dot{m}_0(D)[\alpha - \theta_r], \quad \textit{if} \quad \beta = 0, \qquad (12)$$

which is the same as formula (8).

In Fig. 4 we plot the mass flow rate $\dot{m}_\beta$ (formula (10)), as a function of β, for beach sand. For the sake of simplicity, in Fig. 4 we took into account plots for beach sand emerging from orifices $D = 1.00 \pm 0.05$ cm diameter on face walls $w = 0.3$ and $w = 0.6$ cm wall thicknesses, respectively. In Fig. 4 we immediately observe that for each plot we have the values β^* for which the flow is halted ($\dot{m}_\beta = 0$). These values are $\beta_1^* = -0.460$ for $w = 0.6$ cm (solid curve), $\beta_2^* = -0.709$ for $w = 0.3$ cm (dashed curve) and $\beta_3^* = -1$ (dotted curve) which correspond, ideally, to the case without wall thickness. In Fig. 4 this latest case has also been plotted as a comparison with actual cases where the face walls have a finite wall thickness. We do not show plots for sugar because the behavior of the curves is very similar to that of sand. However, in Table 1, we give values of arrest angles β^* for sand and sugar.

In this stage we observe through the comparison among the experimental and theoretical values of β^*, that there are discrepancies of around 4° (0.069 rad) among the data, *i.e.*, regularly the flow is a halted at lower (negative) values than the theoretically predicted angle, β^*. We believe that perhaps this is due to the wall friction which was not considered in our correlation (10) but in actual cases it is added to the friction of the granular material and counteracts the motion induced by the grain weight. Similar discrepancies in the critical angle of arrest have been reported elsewhere [24].

Inasmuch as our goal is to correlate the experimental measurements of the mass flow rate, $\dot{m}_{\beta\mathrm{expt}}$, with the mass flow rate, $\dot{m}_\beta(D, w, \beta)$, it is necessary to analyze in more detail the behavior of $\dot{m}_\beta$. Thus, we notice that the profiles of $\dot{m}_\beta$ in Fig. 4 reach maximum values at $\beta = \beta_m$ and afterwards the curves fall down. It means that in a $\dot{m}_{\beta\mathrm{expt}}$ *vs.* $\dot{m}_\beta$ plot, the values in the domain ($\dot{m}_\beta$), beyond $\dot{m}_\beta(\beta = \beta_m)$, will also decrease. In order to avoid this we need to extend such a domain up to reach values of β close to $\pi/2$, the cases of orifices on horizontal walls, as we do in the following lines.

The estimation of the β_m value, where $\dot{m}_\beta$ attains a maximum and then changes its slope is computed when $d\dot{m}_\beta/d\beta = 0$, where $\dot{m}_\beta$ is given by relation (10). Doing so, results in

$$\beta + \alpha - \theta_r = -\tan[\beta + \alpha - \theta_r]. \qquad (13)$$

Consequently, to get the β_m value for which $\dot{m}_\beta$ has a maximum we need to solve the transcendental equation $x = -\tan x$, where now $x = \beta_m + \alpha - \theta_r$. The general solution of Eq. (13), gives

$$\beta_m = 2.028 - (\alpha - \theta_r), \qquad (14)$$

and it fixes the largest value at which formula (10) is a monotonic function. In Eq. (14) the value 2.028 is correct up to a numerical precision of

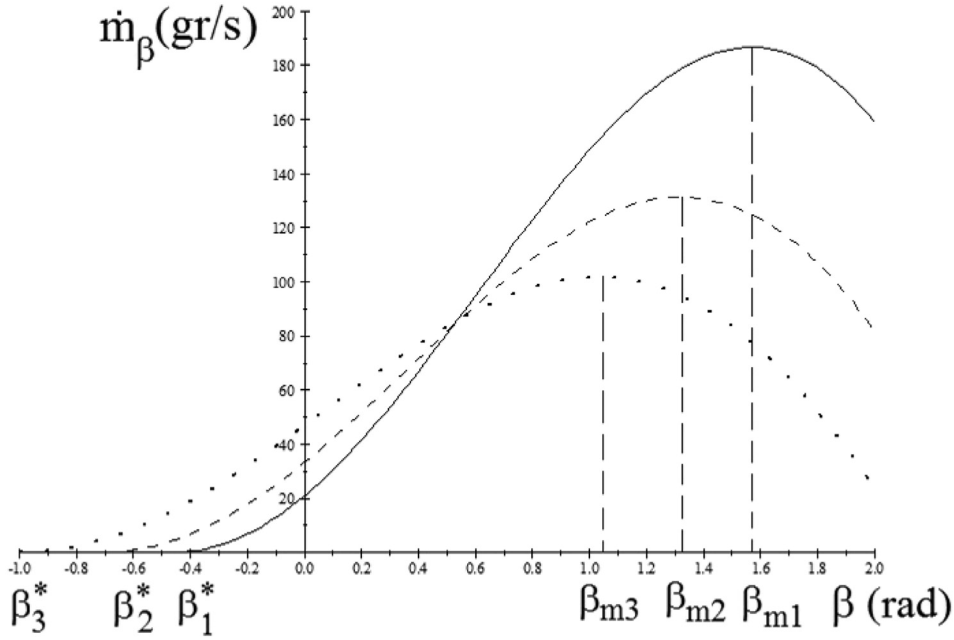

Fig. 4. Plots of the mass flow rates $\dot{m}_\beta$ (formula (10)), as a function of β. Data corresponding to beach sand that outflows through orifices $D = 1.00 \pm 0.05$ cm in diameter, with thicknesses of $w = 0.3$ cm (dashed curve) and $w = 0.6$ cm (solid curve). The dotted curve corresponds to an ideal case where $D = 1.00 \pm 0.05$ cm and $w = 0$ cm (null wall thickness) and was included to contrast with the cases of finite wall thickness. The values β_i^* correspond to the angles of arrest of the granular flow and the values β_{mi} indicate the values of β for which the curve $\dot{m}_\beta$ attains a maximum.

10^{-13}. By using Eq. (14) we computed the values of β_m for sand and sugar. Namely, for sand we found that $\beta_{m1} = 1.567$ if $w = 0.6$ cm, $\beta_{m2} = 1.318$ if $w = 0.3$ cm and finally for the ideal case, when $w = 0$ cm, $\beta_{m3} = 1.028$. In the case of sugar we found that $\beta_{m1} = 1.319$ for $w = 0.9$ cm, $\beta_{m2} = 1.186$ when $w = 0.3$ cm and finally, for the null thickness wall $w = 0$, $\beta_{m3} = 1.037$. Thus, for each couple of values (α, θ_r), depending on the orifice dimensions and the granular material, respectively, we get a value of β_m where the maximum occurs.

Moreover, the use of β_m given by Eq. (14) in formula (10) implies that

$$\dot{m}_\beta(D, w, \beta^*) \propto \frac{1.819\, \dot{m}_0(D)}{\sin(\alpha - \theta_r)}. \tag{15}$$

This result gives the maximum value of $\dot{m}_\beta$ and depends on (α, θ_r) and the corresponding mass flow rate from the horizontal orifice of diameter D.

Consequently, an uniformly valid formula for the complete experimental domain $\beta^* \le \beta \le \pi/2$, is given by the piecewise function

$$\dot{m}_\beta(D, w, \beta) \propto \begin{cases} \dot{m}_0(\beta + \alpha - \theta_r)\dfrac{\sin(\beta + \alpha - \theta_r)}{\sin(\alpha - \theta_r)} & \text{if} \quad \beta^* \le \beta \le \beta_m, \\ \dfrac{1.819\, \dot{m}_0}{\sin(\alpha - \theta_r)}[1 + (\beta - \beta_m)] & \text{if} \quad \beta_m < \beta \le \dfrac{\pi}{2}, \end{cases} \tag{16}$$

Table 1
Values of α, the characteristic angles computed for the orifices of diameter D and thickness w. The angles β^* and β_m are computed by using Eqs. (11) and (14), respectively, for sand and sugar.

D (cm)	w (cm)	α (rad)	β^* (rad)	β_m (rad)
Sand beach				
1.00 ± 0.05	0.6	1.030 ± 0.041	−0.460	1.567
1.00 ± 0.05	0.3	1.279 ± 0.051	−0.709	1.319
1.00	0	π/2	−1.00	1.028
Granulated sugar				
2.00 ± 0.05	0.9	1.147 ± 0.045	−0.568	1.319
2.00 ± 0.05	0.3	1.422 ± 0.056	−0.842	1.186
2.00	0	π/2	−0.990	1.037

where the β values in the region $\beta_m < \beta \le \pi/2$ increase very slightly in respect to the cut off angle, β_m. Formula (16) is a continuous function because for any value in the domain $\beta^* \le \beta \le \pi/2$ there exists a well-defined value of $\dot{m}_\beta(D, w, \beta)$ and small changes in β result in small changes in $\dot{m}_\beta$.

By using the correlation (16), plots of $\dot{m}_{\beta\text{expt}}$ *vs.* $\dot{m}_\beta$, for sand and sugar, are shown in Fig. 5. The best fits to data are straight lines of the form

$$\dot{m}_{\beta\text{expt}} = c\dot{m}_\beta(D, w, \beta), \tag{17}$$

where c is the respective discharge coefficient and $\dot{m}_\beta(D, w, \beta)$ is the piecewise function given by formula (16).

From fits in Fig. 5(a) we found that $c = 0.149 \pm 0.002$ if $w = 0.3$ cm and $c = 0.114 \pm 0.002$ if $w = 0.6$ cm when sand and orifices of $D = 1.00 \pm 0.05$ cm diameter were used. For sugar and orifices $D = 2.00 \pm 0.05$ cm diameter we found that fits in Fig. 5(b) yield $c = 0.185 \pm 0.002$ if $w = 0.3$ cm and $c = 0.231 \pm 0.002$ if $w = 0.9$ cm.

5. Conclusions

In this work we studied experimentally the problem of the mass discharge rate of cohesionless granular material from circular orifices on sidewalls of tilted bins. We started our analysis from the consideration of a gradual (positive and negative) inclination of the bin. Experiments using beach sand and granulated sugar showed that the flow rate behaves as a sine-like function when it is plotted as a function of the tilt angle β. Experiments also showed that for orifices with the same diameter but different wall thicknesses, the mass flow rate attains near the same value at $\beta = \pi/2$, *i.e.*, for orifices in horizontal walls the wall thickness does not influence the mass flow rate, which is a well known result [2]. The experimental results and the existence of a correlation for the mass flow rate of granular material from vertical sidewalls of bins allowed the formulation of a reliable correlation that takes into account the orifice diameter, the wall thickness (which other models consider as irrelevant [2,23,32], despite sidewall thickness is ubiquitous and crucial in the flow occurrence) and the tilt angle. Because of this, the resulting experimental correlation, that relates linearly both the experimentally

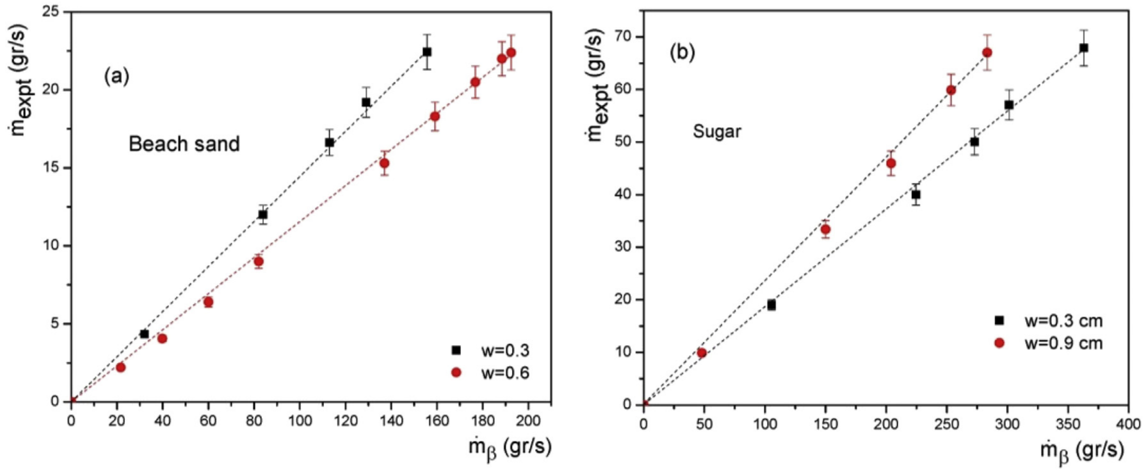

Fig. 5. Plots of $\dot{m}_{\beta\,\mathrm{expt}}$ *vs.* the correlation $\dot{m}_{\beta}$ given by formula (16): in (a) beach sand was used and orifices $D = 1.00 \pm 0.05$ cm in diameter on sidewalls with thicknesses $w = 0.3$ cm and $w = 0.6$ cm, respectively. In (b) we used granulated sugar, $D = 2.00 \pm 0.05$ cm in diameter and with thicknesses of $w = 0.3$ cm and $w = 0.9$ cm, respectively.

measured mass flow rate and the piecewise function, Eq. (17), fulfills several important limits: a) it predicts approximately the value for which the flow must be arrested, $\beta^* = -(\alpha - \theta_r)$, b) it is reduced to the formula of the flow rate from lateral orifices when $\beta = 0$ and c) it allows to predict the cut off angle, β_m, for which the sine-like mass flow rate matches with a linear function of the angle β (second part of the formula (16)) in order to span the overall values of the tilt angle: $\beta^* \le \beta \le \pi/2$. The correctness of the latest formulation (Eq. (17)) is backed by the well fit among $\dot{m}_{\beta\mathrm{expt}}$ and $\dot{m}_{\beta}(D, w, \beta)$. We consider that the small discrepancy in the exact prediction of the arrest angles for the analyzed cases is an important but minor problem in terms that a lot of data meet Eq. (17) and it gives reliability to such an equation. Finally, it seems that the ideal cases where there are orifices on the sidewalls without thickness (or cases with very thin sidewall) can also be correctly estimated with our model.

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