Capillary Rise in a Taylor-Hauksbee Cell with a Tilted Edge

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Abstract In this work we discuss a series of experiments to get the equilibrium profiles when a viscous liquid rises spontaneously in the wedge-shaped gap between two vertical plates intersecting at a tight angle $\alpha \ll 1$. We contrast the differences between the case with vertical edge and those where the aristae is tilted to the vertical. Our theoretical model agrees very well with the experimental data.

1 Introduction

The problem of the spontaneous capillary rise of a liquid into a vertical cell made by two plates touching on an edge and having a small angle among them (wedge) was initially studied experimentally by Taylor (1712) and Hauksbee (1712). In both experiments the equilibrium profiles were rectangular hyperbolas with a very important particularity: near the edge, liquid can reach an infinite height, which for purposes of fractured oil fields it may be a huge problem because due to this dispersion the oil recovery may be limited. Conversely, in the process of feeding of trees this mechanism is very suitable to transport liquids as water and sap (which contains the

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nutrients) because the equilibrium heights attained through the xylem are unlimited and it promotes photosynthesis and transpiration in the leaves.

Historically, for nearly three hundred years the dynamic problem, to the time evolution liquid's free surface, towards the equilibrium, remained unresolved until by 2008 our group published a paper with the solution to this problem (Higuera et al. 2008).

In this work we report a series of experiments where the wedge's arista is tilted with to the vertical. This slope changes the shape of the equilibrium and dynamical profiles with respect to cases with vertical edges. Particularly, we are interested in finding the equilibrium shapes of the free surfaces, theoretically and experimentally. To reach such a goal, this work is divided as follows: in the next section we reviewed the problem of the capillary rise in the Taylor-Hauksbee cell. Later on, in the same section, we proposed a theoretical description of the equilibrium profiles in cells with tilted edges. Experimental data yield that such description is a suitable way to show such profiles. In Sect. 3 we show a set of dynamical profiles. Finally in Sect. 4 we present the main conclusions for this work.

2 Capillary Rise in the Taylor-Hauksbee Cell

2.1 Cell with a Vertical Edge

The wedge-shaped gap between two vertical plates intersecting at an angle $\alpha \ll 1$ is initially empty. At a certain moment, the lower edges of the plates get in touch with a liquid of density ρ, dynamic viscosity μ, and surface tension σ. The liquid wets the plates with a contact angle $\theta < \pi/2$ and therefore rises between the plates by capillary action as shown in Fig. 1a. The ratio of the two principal curvatures of the free surface of the liquid between the plates is small, on the order of α. The normal section of maximum curvature, by a plane nearly normal to the plates, is approximately a circular arc of radius $\alpha x/2\cos\theta$, where x is the distance to the line of intersection of the plates. The pressure jump across the surface is approximately

$$\Delta p = \frac{2\sigma\cos\theta}{\alpha x}. \tag{1}$$

At equilibrium, the height $H_e(x)$ of the meniscus above the level of the outer liquid is determined by the balance

$$\Delta p = \rho g H_e, \tag{2}$$

where g is the acceleration due to gravity. This balance gives the rectangular hyperbola (Concus and Finn 1969)

$$H_e = \frac{2\sigma\cos\theta}{\rho g \alpha x}. \tag{3}$$

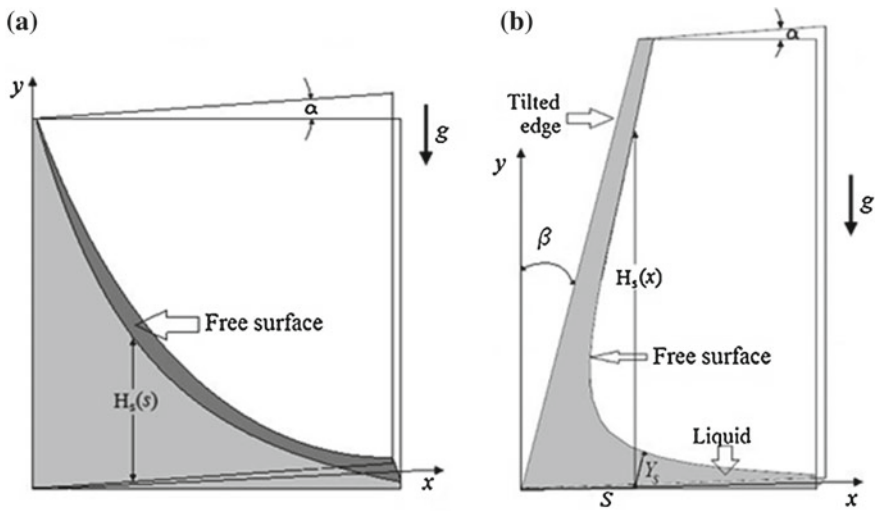


Fig. 1 **a** Scheme of the Taylor-Hauksbee cell. In this case the equilibrium profiles are given by $H_S(x)$. **b** Taylor-Hauksbee cell with a tilted edge. In this case Y_s is a distance to the equilibrium profile along the line parallel to the tilted edge and that starts at x = s and y = 0; Thus the most suitable coordinate system is given by (s, Y_s)

In order to compare the latter theoretical profile with experiments, we performed a series of experiments with silicon oil of viscosity $\mu = 100$ cP, surface tension $\sigma = 0.0215\,\mathrm{N/m}$, density $\rho = 971\,\mathrm{kg/m^3}$ and an aperture angle of $\alpha = 0.0166\,\mathrm{rad}$. In experiments we used a digital Cannon Réflex T3i camera to take pictures and a video recording of the capillary rise. In Fig. 2 we show a picture with the equilibrium profile and also a comparison between the experimental data and the theoretical profile given by Eq. (3). The fit is very good assuming that $\theta = 0\,\mathrm{rad}$.

2.2 *Cell with Tilted Edge*

In this part we consider the cases of cells with tilted edges with respect to the vertical, see Fig. 1b. The usual Cartesian coordinates appears no suitable to depict the equilibrium profiles because it is possible that for some values of x there are two values of $H(x)$ (the equilibrium profile). Thus, the concept of function can be lost.

Instead, we choose the rotated coordinate system (x', y') to describe the equilibrium profile. We analyze the problem for the point (s, Y_s) shown in Fig. 1b. s is the distance from the lower apex $(x' = 0, y' = 0)$ to any point along x, whereas Y_s is the distance to the equilibrium profile along the line parallel to the tilted edge (y') and that starts at the point $x' = s\cos\beta$ and $y' = s\sin\beta$; thus the equilibrium profile is given by the injective function $y'(x')$.

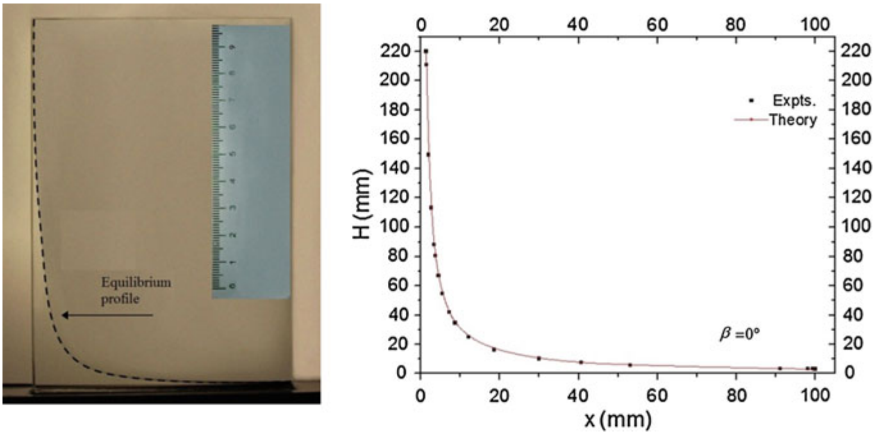


Fig. 2 *Left hand side* picture of the experimental equilibrium profile (rectangular hyperbola) for $\beta = 0$ rad. *Right hand side* theoretical profile (*curve*) and experimental data (*symbols*)

In this case the pressure jump across the surface is approximately

$$\Delta p = \frac{2\sigma \cos\theta}{\alpha s \cos\beta}, \tag{4}$$

meanwhile the equilibrium profile is determined by the balance

$$\Delta p = \rho g Y_s \cos\beta. \tag{5}$$

This balance gives the hyperbola

$$Y_s = \frac{2\sigma \cos\theta}{\rho g \alpha s \cos^2\beta}, \tag{6}$$

Another way to get the profile is by using the vertical length $H_s(s)$ but it is more complex because for some values of s could exist two values of H_s and for other ones, none. Thus, it is better to compute $s(H_s)$ (an injective function) given by

$$S = \frac{2\sigma \cos\theta}{\rho g \alpha \cos\beta H_s} + H_s \tan\beta. \tag{7}$$

Consequently, the equilibrium profile can be plotted by using Eqs. (6) or (7).

To make sure that our model is correct in the prediction of the equilibrium profiles in Fig. 3 we show the actual equilibrium profile in a Taylor-Hauskbee cell having $\beta = 0.523$ rad (30°) and $\alpha = 0.0092$ rad. The liquid of work was again silicone oil of $\mu = 100$ cP. In Fig. 4 we give an example of Fig. 3 showing other details.

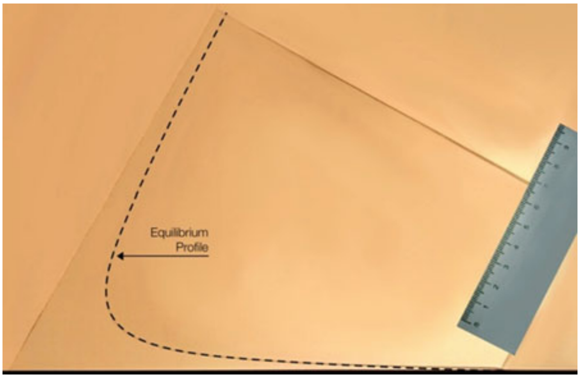


Fig. 3 Picture of the experimental equilibrium profile for $\beta = 0.523$ rad (30°) and $\alpha = 0.0092$ rad

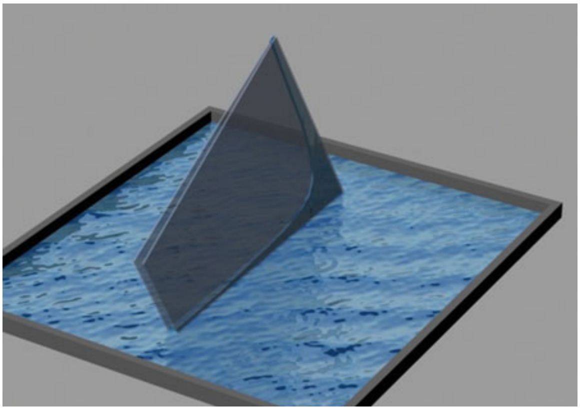


Fig. 4 Depict of the experimental equilibrium profile showing other details of the cell of Fig. 3

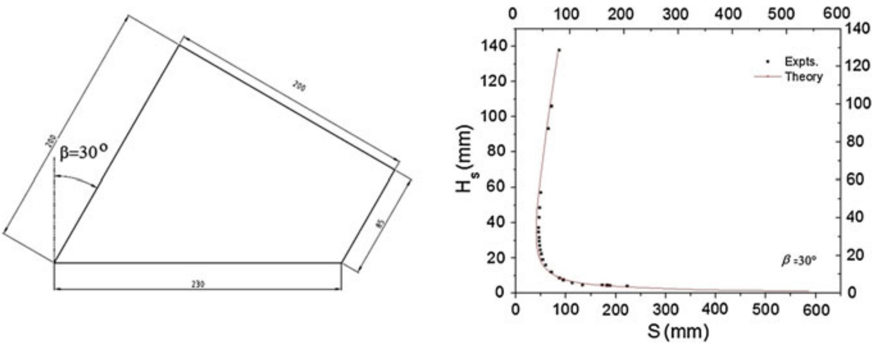


Fig. 5 *Left hand side* Scheme of the cell with its dimensions (in mm) for $\beta = 0.523$ rad (30°). *Right hand side* theoretical profile (*curve*) and experimental data (*symbols*), here $\alpha = 0.0092$ rad

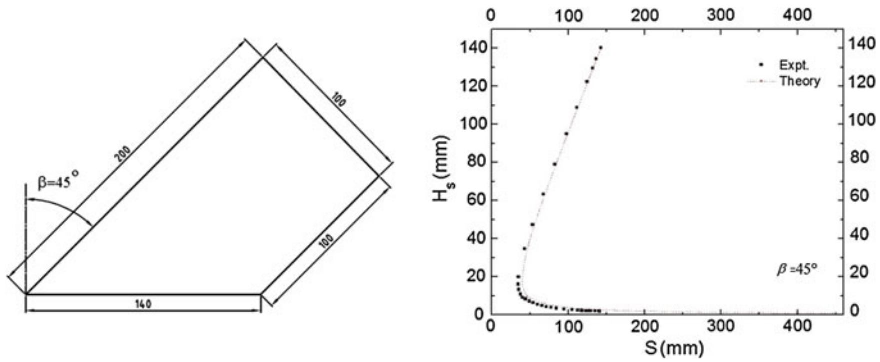
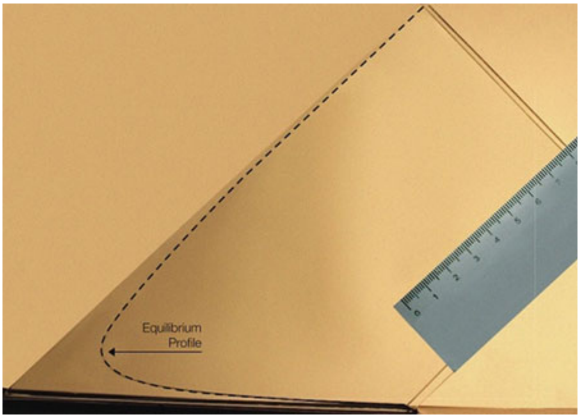


Fig. 6 *Left hand side* Scheme of the cell with its dimensions (in mm) for $\beta = 0.785$ rad (45°). *Right hand side* theoretical profile (*curve*) and experimental data (*symbols*), here $\alpha = 0.01659$ rad

Fig. 7 Picture of the experimental equilibrium profile for $\beta = 0.785$ rad (45°) and $\alpha = 0.0166$ rad



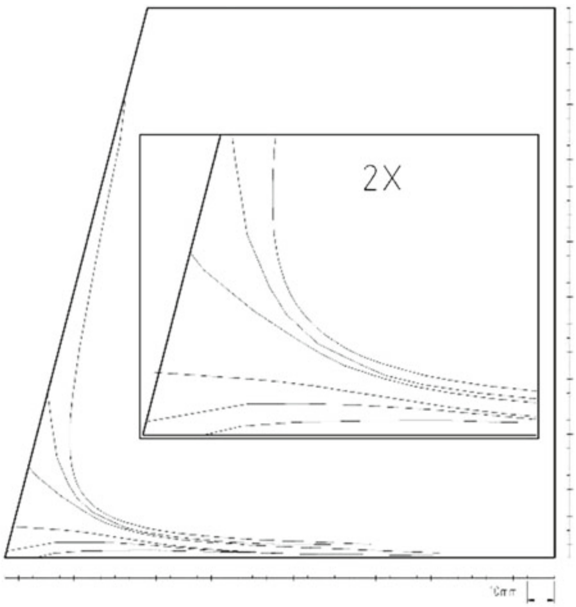
The actual dimensions of the cell are given in Fig. 5. There, we also show the plot of the theoretical profile Eq. (7) as well as the experimental data obtained from Fig. 3. The fit is very fine.

In Fig. 6 we show a scheme of the cell and the plot of the actual equilibrium profile for $\beta = 0.785$ rad (45°) and $\alpha = 0.0166$ rad. Data were taken from Fig. 7. Clearly the fit is very good.

3 Dynamical Evolution

The equilibrium profile is attained after a succession of free surfaces. In Fig. 8 we show a plot with several free surfaces obtained at several time instants. Such a behavior is very similar to that occurring in the normal Taylor-Hauksbee cell (Higuera et al. 2008). In spite of this result, the times to reach the equilibrium between the

Fig. 8 Plot with the profiles at several stages (time instants) towards the equilibrium. In this case $\beta = 0.523$ rad (30°) and $\alpha = 0.0092$ rad



case with vertical edge and those where the edges are tilted was very different from each other. This is due to the intensity of the gravity field which changes with β.

4 Conclusions

In this work we have analyzed the problem of the capillary rise of a viscous liquid into vertical and tilted Taylor-Hauskbee cells. We have put forward a theoretical model to describe the actual equilibrium profiles. The model predicts pretty well the profiles in such a cell. The transition to the equilibrium profiles shows complex instantaneous profiles very similar to those occurring in a vertical cell. Further work along this line is in progress.

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