Outflow of granular material from orifices of thin vertical sidewalls

D.A. Serrano, J. Klapp

Departamento de Matemáticas, Cinvestav del Instituto Politécnico Nacional., Av. Instituto Politécnico Nacional 2508, Col. San Pedro Zacatenco, Delegación Gustavo A. Madero 07360, CDMX México.

A. López-Villa, A. Medina*

ESIME Azcapotzalco, Instituto Politécnico Nacional. Av. de las Granjas 682, Col. Sta. Catarina, Azcapotzalco 02250, CDMX México.

C.A.Vargas

Laboratorio de Sistemas Complejos, Departamento de Ciencias Básicas, UAM Azcapotzalco, Av. San Pablo 180, Azcapotzalco 02200, CDMX Mexico.

Abstract

In this work we report experiments carried out to describe the features of the mass flow rate of dry cohesionless granular material that flows, only due to gravity, out from circular orifices at the sidewall of silos with thin wall thickness, w. This study shows that the classical Hagen's correlation for the mass flow rate is essentially correct for this case.

PACS: 45.70.Mg; 45.70.-n

Keywords: Classical mechanics of discrete systems; Classical mechanics of granular systems

* Corresponding author; E-mail: amedinao@ipn.mx

I. INTRODUCTION

The discharge of non cohesive granular material from the bottom exit of containers, due only to gravity action, is a very interesting phenomenon because it is independent of filling depth and non linearly dependent on the orifice size. The first fundamental proposal of a valid formula for the flow rate of grains emerging from the bottom, $\dot{m}_0$, was developed by Gotthilf Hagen [1] a German hydraulic engineer who in 1852 discovered experimentally a relationship of the form: $\dot{m}_0 = \rho g^{1/2} D^{5/2}$, where ρ is the bulk density, g is the acceleration due to gravity and D is the diameter of the circular orifice. Notoriously, this formula constitutes the foundation of the hourglass theory [2–4]. Industrially, it is the basis to estimate controlled granular flows as dosage of powders and granules [5, 6], among others.

Despite the enormous utility of Hagen's correlation, only a few studies have been conducted to test its validity in the discharge of grains through orifices on the vertical sidewalls of bins [7–14]. To the best of our knowledge, Franklin and Johanson [7] were the first researchers who studied experimentally the problem of the gravity driven lateral outflow of granular material from boxes. In their studies, they used galvanized sheet metal 0.238 cm thickness and they found that the flow rate, for orifices at the vertical sidewall, scales as $\dot{m} = c\rho g^{1/2} D^{5/2}$ where c is a factor dependent on the angle of repose of the granular material, θ_r. In such work, no consideration of the wall thickness was invoked. Later, Bagrintsev and Koshkovskii [8] used oval and circular exit holes, made on the transparent plastic walls of verticals cylinders whose thicknesses were 0.1 and 0.45 cm. They found that the mass flow rate scales approximately as $\dot{m} = c\rho g^{1/2} D^{7/2}$ where now c is a parameter that depends on the geometrical features of the exit hole; they also qualitatively observed that "to obtain the greatest possible outflow capacity through a circular hole in a cylindrical tube it is necessary to use a tube of minimum allowable wall thickness". Later, Chang *et al* [9] and Davies and Foye [10] carried out experiments in rectangular vessels with rectangular exit slots. Chang *et al* did not report the dimensions of the wall thickness of their containers and Davies and Foye [10] have reported the use of mild steel sheets 0.12 cm thick. In spite of it, all of these authors essentially found that the mass discharge obeys the relation $\dot{m} = \rho g^{1/2} D^{5/2}$, as Sheldon and Durian [10], who used circular exit holes on steel cans 0.025 cm wall thickness and square Aluminum tubings 0.03 cm wall thicknesses. Nevertheless, none of the previously referred works analyzed the effect of the wall thickness on $\dot{m}$ systematically.

Recently, we have found [12–14], by means of experiments, a general correlation for the flow rate of granular material through orifices in the sidewalls that involves the important effect of the wall thickness, w. Such a general formula predicts that when the wall thickness is very small, the Hagen's correlation must *necessarily* be valid. We get this result by assuming, as a particular case, the consideration of negligible thickness in our formula of the mass discharge from sidewalls of any thickness. Systematic experiments reported here will validate such prediction.

It is important to comment that from the point of view of the structural design, the condition of a very thin wall is inconvenient in actual industrial silos, due to the elastic deformation of a cylindrical silo (in the limit of small deformations) changes essentially as $1/(Ew)$ [15], where E is the elasticity modulus of the material of the wall and consequently a very thin sidewall ($w \to 0$) or a very small resistance of the wall that is deformed elastically ($E \to 0$) may promote permanent deformation of the silo and/or structural damage.

To reach our goal, the plan of this work is as follows. First, in next Section we will analyze, through our formula [12–14], the main results to estimate the mass flow rate from orifices with very thin lateral walls. Then, in Section 3, we are going to report experiments of discharge rates in this limit case. We found that the experiments fit a Hagen-like correlation valid when $w \to 0$ pretty well. All those results will allow us, in Section 4, to give a geometrical and physical interpretation of the mass flow rate formula for any thickness w, including the case when $w = 0$. Finally, in the last Section we will give the main conclusions of the study here tackled.

II. NEGLIGIBLE THICKNESS LIMIT

As aforementioned, the wall thickness affects the granular flow when it occurs across sidewalls, *i.e.*, w also modulates $\dot{m}$, the lateral outflow, provided $D/d_g >> 1$ [3]. In Fig. 1, we depict a scheme that represents the transversal region close to a hole of size D in a vertical sidewall of the container. In Fig. 1(c) becomes evident that there is always a natural *angle of wall*, α, which can be defined as $\alpha = \arctan(D/w)$. Meanwhile, in this same figure it is observed that if there is no flow, due to the wall thickness being wide enough, the granular material contained there will attain its angle of repose. Thus, an outflow is kept as long as $\alpha > \theta_r$ (see Fig. 1(a) and 1(b)) and, conversely, the outflow should be arrested if

$\theta_r \geqslant \alpha$. Consequently, the mass flow rate itself must be proportional to $(\alpha - \theta_r)$.
Another important feature to get a general relation for $\dot{m}$ is that the mass flow rate through vertical holes is a fraction of $\dot{m}_0$ (the mass flow rate through bottom holes) [7, 11]. Therefore, the mass flow rate dependent on D and w is a relation of the form [12–14]

$$\dot{m} = c\dot{m}_0[\alpha - \theta_r] = c\dot{m}_0 \left[\arctan\left(\frac{D}{w}\right) - \theta_r\right]. \tag{1}$$

where c is the dimensionless discharge parameter corresponding to the lateral outflow and $\dot{m}_0$ is given by the Hagen's correlation

$$\dot{m}_0 = \rho g^{1/2} D^{5/2}. \tag{2}$$

A set of experimental studies have shown that the formula (1) describes the mass flow rate of granular solids crossing a vertical wall when $w \sim D$ [12–14] pretty well.

A simpler formula for the lateral flow rate across thin-walled silos can be obtained from (1) expanding in powers series the function $\arctan(D/w)$, when $w \to 0$, and D is a valid diameter, to first order it yields

$$\dot{m} \approx c\dot{m}_0[\frac{\pi}{2} - \theta_r - \frac{w}{D}] \text{ for } D \gg w, \tag{3}$$

therefore, as a direct consequence of (3), if the wall thickness is negligible (formally $w = 0$) we get that

$$\dot{m} \approx c\rho g^{1/2} D^{5/2}[\frac{\pi}{2} - \theta_r], \tag{4}$$

where we have used the relationship (2). The correlation (4) expresses that the mass flow rate for the lateral outflow essentially obeys the Hagen formula because the factor $(\pi/2 - \theta_r)$ is a numerical quantity, independent of D and w. It confirms the experimental results reported by several authors [7, 9–11] who worked with thin face walls. Thus, the Hagen correlation results valid when the orifices are made on a very thin sidewall.

III. EXPERIMENTS

In order to compare the behavior of the mass flow rate for both, bottom exits and lateral exits, we performed experiments with a steel can 13 cm diameter, 22 cm height and

$w = 0.0080 \pm 0.0005$ cm thickness. In this vessel we drilled four circular orifices spatially distributed at the bottom and four orifices equally distributed around the silo perimeter and 8 cm above the bottom. The diameters of the holes were $D = 0.50 \pm 0.05, 0.70 \pm 0.05, 1.10 \pm 0.05$ and 1.70 ± 0.05 cm. We use beach sand (mean diameter $d_g = 0.03$ cm, bulk density $\rho = 1.5$ gr/cm^3 and angle of repose $\theta_r = 33^o \pm 0.5^o = 0.57 \pm 0.008$ rad) and granulated sugar (mean diameter $d_g = 0.073$ cm, bulk density $\rho = 0.84$ gr/cm^3 and angle of repose $\theta_r = 33.5^o \pm \pm 0.5^o = 0.58 \pm 0.008$ rad) as granular solids. The section of the laboratory in which the experiments were made was climate controlled (25 ± 1 °C and 45 ± 10% R.H.). The moisture contents of sand and granulated sugar samples were 0.50 ± 0.06% and 0.015 ± 0.005% w.b., respectively. Details of the measurement procedure of the discharge rates by using a force sensor are given elsewhere [12–14].

In the first stage of the experiments we measured the mass flow rate of both bulk solids. In Fig. 2 we show the plot of the experimental flow rate, $\dot{m}_{Expt}$, as a function of D, for bulk materials outflowing from bottom and lateral exits. Figure 2(a) corresponds to sand and Fig. 2(b) is for sugar. For both materials the mass flow rates behave essentially as

$$\dot{m}_{Expt} \propto D^{5/2}. \tag{5}$$

Moreover, in Fig. 3 we give plots of the respective sand and sugar experimental mass flow rates, $\dot{m}_{Expt}$, versus $\rho g^{1/2} D^{5/2}$ (for bottom exits) and $\rho g^{1/2} D^{5/2}[\frac{\pi}{2} - \theta_r]$ (for lateral holes). It can be easily appreciated that the best fit for data obeys a linear relation

$$\dot{m}_{Expt} = \left\{ \begin{matrix} a\rho g^{1/2} D^{5/2}, \text{ for bottom exits} \\ c\rho g^{1/2} D^{5/2}[\frac{\pi}{2} - \theta_r], \text{ for lateral exits} \end{matrix} \right\}, \tag{6}$$

where the dimensionless discharge coefficients have the value $a = 0.48 \pm 0.01$ for sand and sugar emerging from bottom exits. Similarly, the discharge coefficients for lateral outflow have approximately the same value $c = 0.16 \pm 0.01$ for both materials. Incidentally, the ratio of vertical to horizontal discharge rates is around one third, which is close to values reported in experiments using agricultural grains [9]. All the latest results allow us to prove that the Hagen's correlation describe fine the mass flow rate in both configurations.

IV. GEOMETRICAL APPROACH OF THE MASS FLOW RATE CORRELATION

From a geometrical point of view, formula (1), which is valid for the flow rate of lateral outflow, represents a three-dimensional surface; by using the correlation (2) in such a formula we have that its most explicit form is given by the expression

$$\dot{m}_{Expt}(D,w) = c\rho g^{1/2} D^{5/2} \left[\arctan\left(\frac{D}{w}\right) - \theta_r\right]. \tag{7}$$

In Fig. 4 we show, for sand ($c = 0.16 \pm 0.01$), two different views of the surface $\dot{m}_{Expt}(D,w)$ which may correspond to actual values of the mass flow rate for the couple values (D,w), where necessarily $D > 6d_g$ (in order to avoid a possible clogging or arrest of the granular flow [3]) and $w \geq 0$.

The surface on the left hand side of Fig. 4 show the general non linear behavior of correlation (7) and in the plot on the right hand side we show, as a part of the surface, its intersection with the plane $w = 0$, which describes the curve $\dot{m}(D,0) = c\rho g^{1/2} D^{5/2}[\pi/2 - \theta_r]$ given in (6) and part of which was depicted in Fig. 2(a). In this particular case, there is a granular flow for any value in which $D > 6d_g$.

We also observe that the three-dimensional surface of Fig. 4 shows graphically that if w and D are similar the flow is weak, whereas if $D \gg w$, then the granular flow is strong. Thus, the maximum flow rate occurs when $w \to 0$ (actually a very thin sidewall).

A fundamental fact of the surface is that a mass flow rate reaches a given value, say $\dot{m}(D,w)$, only for a specific couple of values of (D,w): mathematically it means that $\dot{m}(D,w)$ is an injective function or one-to-one function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain. Clearly, the error inherent to the experiments may imply a slight "thickness" of such a surface.

V. CONCLUSIONS

This study completes the formulation of a correlation to accurately describe the mass flow rate for lateral and circular exit holes of diameter D and wall thickness w. In particular, here we demonstrated that the Hagen formula, where $\dot{m} \propto D^{5/2}$, is suitable to quantify the flow rate across orifices on the sidewall of containers with very thin vertical walls. We also have shown that the geometrical representation of $\dot{m}(D,w)$ allows to define several important characteristics of the flow. Finally, we believe that this type of studies allow us to reach a deeper understanding of the mechanics of the discharge of non cohesive granular materials.

VI. ACKNOWLEDGEMENTS

Authors acknowledge partial supports of projects 20151924 and 20160942, both from Secretaría de Investigación y Posgrado, IPN.

---

[1] G.H.L. Hagen, Bericht uber die zur Bekanntmachung geeigneten Verhandlungen der Koniglich Preussischen Akademie der Wissenschaften zu Berlin, 35 (1852) 35.

[2] K. Wieghardt, Ann. Rev. Fluid Mech. 7 (1975) 89.

[3] P.G. de Gennes, Physica A 261 (1998) 267.

[4] T. Le Pennec, M. Ammi, J.C. Messager, B. Truffin, D. Bideau, J. Gamier, Powder Tech. 85 (1995) 279.

[5] C.R. Woodcock, J. S. Mason, Bulk solids handling: and introduction to the practice and technology, Blackie and Son Ltd, London (1987).

[6] D. McGlinchey (Ed.) Bulk Solids Handling: Equipment Selection and Operation, Blackwell Publishing Ltd (2008).

[7] F.C. Franklin, L.N. Johanson, Chem. Eng. Scie. 4 (1955) 119.

[8] I.I. Bagrintsev, S.S. Koshkovskii, Jour. Chem. Petroleum Eng. 6 (1977) 503.

[9] C.S. Chang, H.H. Converse, J.L. Steele, Trans. Am. Soc. Agr. Eng. 34 (1991) 1789.

[10] C.E. Davies, J. Foye, Trans. Inst. Chem. Engrs. 69 (1991) 369.

[11] H.G. Sheldon, D.J. Durian, Granular Matter 12 (2010) 579.

[12] A. Medina, D.A. Serrano, G.J. Gutiérrez, K. Kesava Rao, C.A. Vargas, Rev. Mex. Fis. 59 (2013) 287.

[13] A. Medina, D. Cabrera, A. López-Villa, M. Pliego, Powder Tech. 253 (2014) 270.

[14] D.A. Serrano, A. Medina, G. Ruíz-Chavarria, M. Pliego, J. Klapp, Powder Tech. 286 (2015) 438.

[15] A. Medina, M. Pliego, F.J. Higuera, Rev. Mex. Fís. 62 (2016) 89.

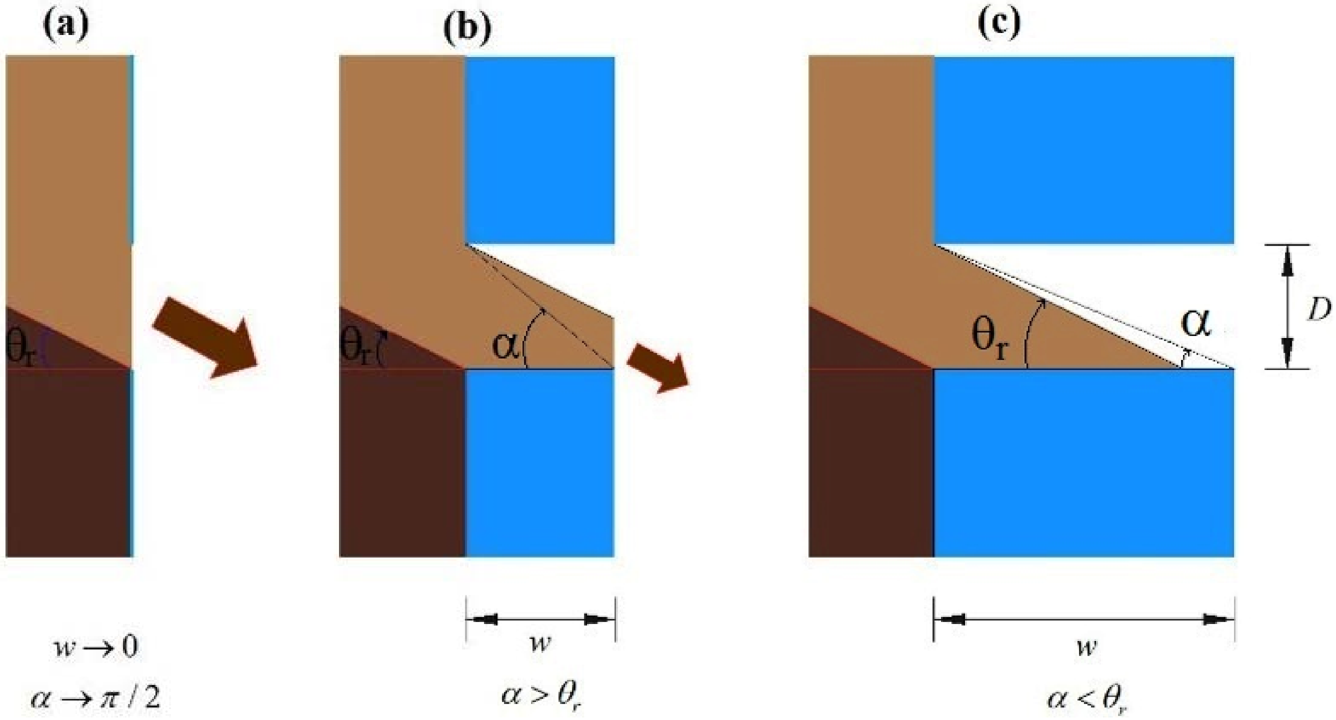
Figure captions

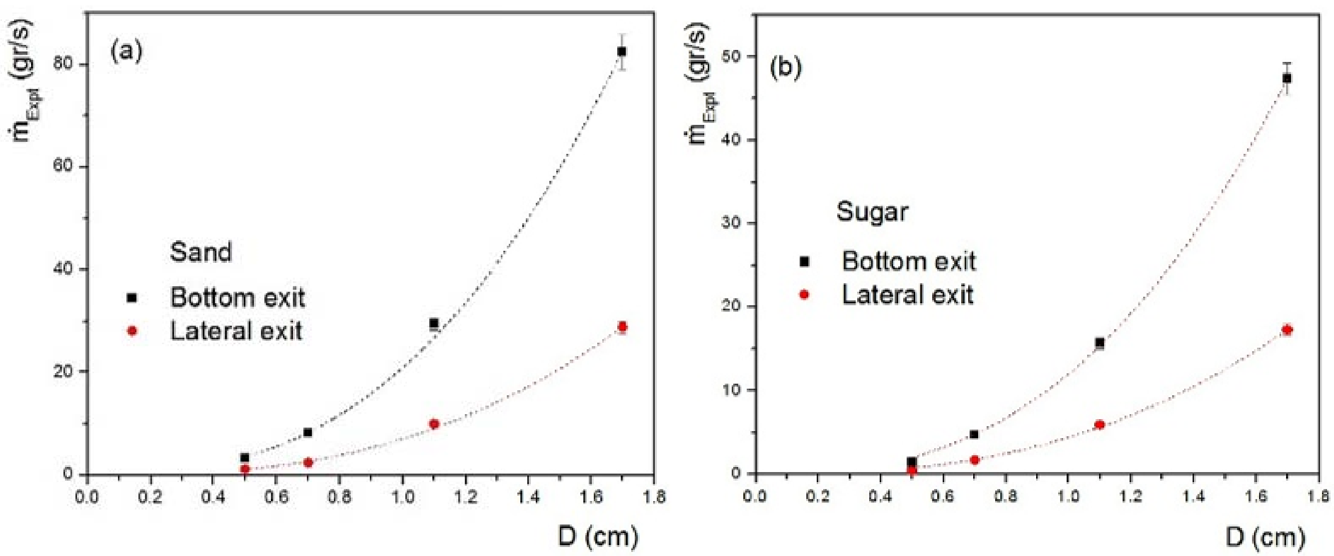
Figure 1, Schematic of the zone close to the vertical exit hole (transversal view): (a) very thin sidewall, there the granular outflow is strong, (b) no negligible wall thickness, the outflow is weak and (c) no flow or arrest flow condition. The exact condition to arrest the flow is $\alpha = \arctan(D/w) \geq \theta_r$ [12–14].

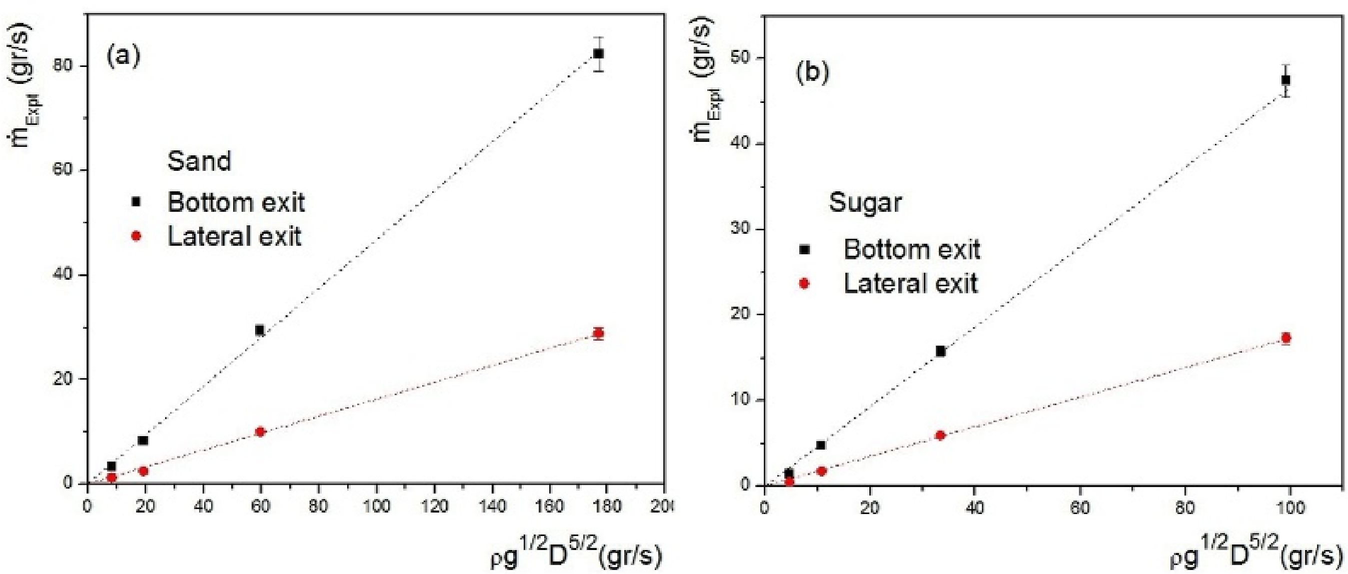
Figure 2, Plot of the experimental mass flow rate $\dot{m}_{Expt}$ as a function of the orifice diameter, D, for bottom exit holes (■) and for orifices in the sidewall (•). (a) Beach sand and (b) granulated sugar. All data fit $\dot{m}_{Expt} \propto D^{5/2}$, error bars are of 4%.

Figure 3, Plots of $\dot{m}_{Expt}$ versus $\rho g^{1/2} D^{5/2}$ (for bottom exit holes and lateral exit holes) because $[\pi/2 - \theta_r] = 1$ and 0.99 for sand and sugar, respectively. (a) experiments using sand and (b) with sugar. The straight lines correspond to linear fits, errors bars are of 4%.

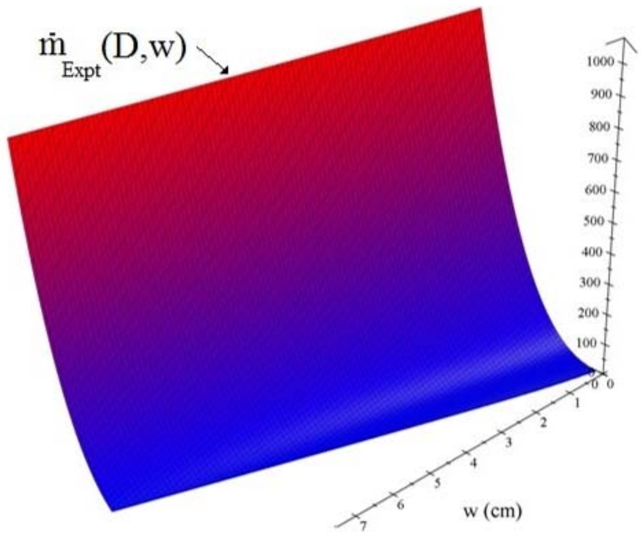
Figure 4, Three-dimensional graphic representation of the actual mass flow rate for sand $\dot{m}_{Expt}(D, w) = c\rho g^{1/2} D^{5/2} [\arctan(D/w) - \theta_r]$. It is important to observe that the 3D surface is valid for any set of values (D,w). In this plot w can be null ($w = 0$) but D must be larger than $6d_g$. When $w = 0$ the mass flow rate is $\dot{m}_{Expt}(D, 0) = c\rho g^{1/2} D^{5/2}$ because in experiments with sand $(\pi/2 - \theta_r) = 1$.


(a)
(b)
(c)
θr
θr
α
θr
α
D
w
w
$w \to 0$
$\alpha \to \pi / 2$
$\alpha > \theta_r$
$\alpha < \theta_r$

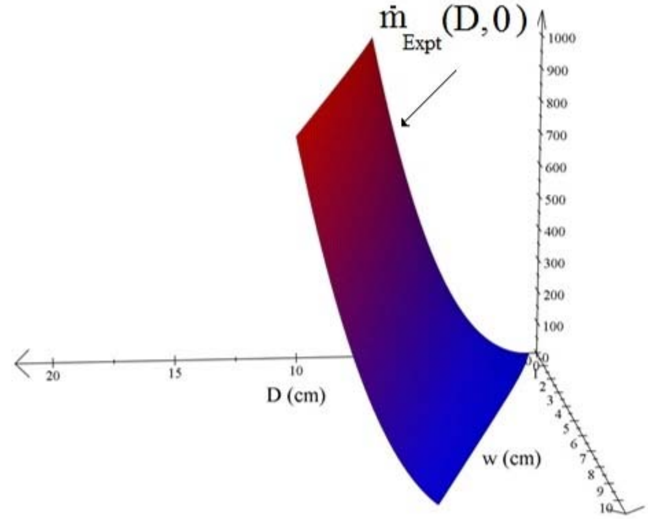

(a)
ṁExpl (gr/s)
Sand
Bottom exit
Lateral exit
D (cm)
0
20
40
60
80
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
(b)
ṁExpl (gr/s)
Sugar
Bottom exit
Lateral exit
D (cm)
0
10
20
30
40
50
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8



(a)
Sand
Bottom exit
Lateral exit
$\dot{m}_{Expl}$ (gr/s)
$\rho g^{1/2}D^{5/2}$(gr/s)
(b)
Sugar
Bottom exit
Lateral exit
$\dot{m}_{Expl}$ (gr/s)
$\rho g^{1/2}D^{5/2}$(gr/s)



$\dot{m}_{Expt}(D,w)$
1000
900
800
700
600
500
400
300
200
100
0
1
2
3
4
5
6
7
w (cm)



$\dot{m}_{Expt}(D,0)$
1000
900
800
700
600
500
400
300
200
100
0
20
15
10
D (cm)
w (cm)
1
2
3
4
5
6
7
8
9
10