Super free fall of a liquid layer in a semi–infinite cone

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In this work the super free fall problem of a very low viscosity mass of liquid which fills partially a section of a long vertical conical pipe is analyzed theoretically, by means of the use of an inviscid and one-dimensional model, the simultaneous and peculiar motion of both two liquid interfaces when the liquid layer is moved from the rest is described.

1. Introduction

Building in the original analysis of Paterson,[1)] it has been recently shown that the upper free surface of a liquid column filling a vertical pipe of downward increasing radius reaches super free fall.[2–4)] It was demonstrated that when a liquid column filling a pipe shaped as an inverted cone of overall opening angle is suddenly released by abruptly opening the lower end of the pipe, the surface of the liquid experiences an initial stage super gravitational acceleration followed by a stage of a sub-gravitational acceleration and a final acceleration increase to exactly gravitational at the end of the discharge.[3)] In the case of pipes with abrupt changes of section (interconnected pipes of different radii), the liquid surface can achieve persistent accelerations several times larger than the gravity acceleration g, and the acceleration is larger from smaller levels of filling in the upper pipe.[4)] This rich dynamics of confined systems contrast with that is observed when an initially confined mass of liquid is suddenly released to the ambient, as in the rupture of a water-filled rubber-ballon.[5)] Then the mass of liquid falls initially following the law of gravitation until disintegrates into smaller droplets. The super free fall of a liquid also resembles the purely rigid body mechanical problem of super free fall of the tip of a chain released from the rest under gravity, either when the chain is initially hanging vertically or when it is folded horizontally.[6)]

In this work we analyze the fall of a finite mass of liquid initially at rest in an infinitely long inverted conical tube. The main difference with the configurations of[2–4)] is that now the liquid never leaves the tube. It is bounded by upper and lower surfaces whose evolution must be described. However, since an algebraic relation exists between the mean levels of the two surfaces, expressing the condition that the volume of liquid between them is constant, it suffices to describes the evolution of one of the surfaces. This is done with the approximations of inviscid quasi-unidimensional flow by means of an evolution equation derived in the spirit of Patterson's analysis,[1)] though the result differs of Patterson's equation owing to the different boundary condition of our problem, and are due to the correction discussed in.[3)] The derivation is presented by a conical tube, but the extension to other cross-section is straightforward. Results of an experimental visualization of the evolution of ethanol in a conical tube with an opening angle of 3° are presented and compared with the numerical results.

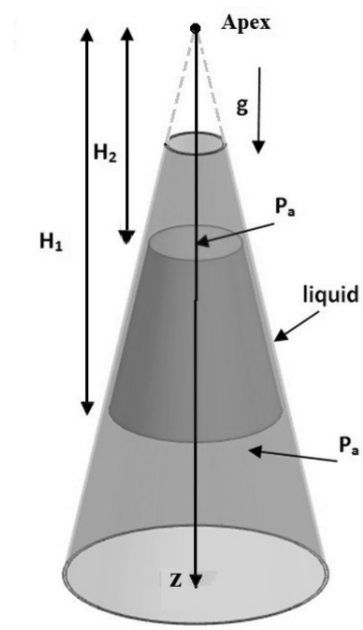

Fig. 1. Scheme of a semi-infinite vertical conical pipe containing a finite volume of an inviscid liquid. The initial position of both lower and upper interfaces are $H_{2(0)}$ and $H_{1(0)}$, respectively..

2. Theory

Consider a long vertical tube shaped on an inverted cone of semi-angle $\theta \ll 1°$. The tube is open at both ends and

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contains a volume V of a liquid which is initially at rest and extends between two horizontal section at distance $H_{2(0)}$ and $H_{1(0)} > H_{2(0)}$ from the vertex of the cone, as sketched in Fig. 1. The quasi–unidirectional motion of a liquid in the tube obeys the mass and momentum conservation equations.

$$\frac{1}{z^2}\frac{\partial}{\partial z}\left(z^2 u\right) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z}\right) = \frac{\partial P}{\partial z} + \rho g, \tag{2}$$

where z is the vertical distance measured downward from the vertex of the cone, t is the time, u and P are the velocity and pressure of the liquid, ρ is the density of the liquid, g is the acceleration due to gravity, and viscous effects has been neglected. Let $H_1(t)$ and $H_2(t)$ denote the advance from the lower and upper surfaces of the liquid from the vertex cone. Then $u = dH_1/dt$, $P = P_a$ at $z = H_1$, $u = dH_2/dt$, $P = P_a$ at $z = H_2$,

$$u = 0, \quad H_1 = H_{1(0)}, \quad H_2 = H_{2(0)} \quad \text{at} \quad t = 0 \tag{3}$$

where P_a is the pressure outside the liquid and surface tensions has been neglected. Equation (1) can be immediately integrated to give, with the condition (3) at the liquid surface,

$$u = \frac{dH_1}{dt}\frac{H_1^2}{z^2} = \frac{dH_2}{dt}\frac{H_2^2}{z^2}. \tag{4}$$

From the second equality, after using the initial condition (3),

$$H_1^3 - H_2^3 = H_{1(0)}^3 - H_{2(0)}^3, \tag{5}$$

which expresses the condition of conservation of the liquid volume. Carrying (4) to the momentum equation (2), integrating the resulting equation from $z = H_1$, to $z = H_2$ and using the boundary conditions (3) for the pressure, we find, after some algebra

$$\frac{d^2H_2}{dt^2} = \frac{H_1}{h_2}g + \frac{1}{2}\left(\frac{dH_2}{dt}\right)^2\left(\frac{1}{H_1} + \frac{H_2}{H_1^2} + \frac{H_2^2}{H_1^2} - \frac{3}{H_2}\right). \tag{6}$$

Equations (5) and (6) determine the function $H_1(t)$ and $H_2(t)$. After these functions are computed, Eq. (4) gives the velocity of the liquid, while the pressure is given by

$$\frac{P - P_a}{\rho} = g\,(z - H_2) + \frac{1}{2}\left(\frac{dH_2^2}{dt}\right)^2\left(1 - \frac{H_2^4}{z^4}\right) + \frac{1}{3}\frac{d^2H_2^3}{dt^2}\left(\frac{1}{H_2} - \frac{1}{z}\right). \tag{7}$$

which is obtained by integrating (2) between $z = H_2$ to a generic value of z. Introducing the dimensionless variables

$$\xi = \frac{H_2}{H_{1(0)} - H_{2(0)}}, \qquad \eta = \frac{H_1}{H_{1(0)} - H_{2(0)}},$$

$$\tilde{z} = \frac{z}{H_{1(0)} - H_{2(0)}}, \qquad \tau = \sqrt{\frac{g}{H_{1(0)} - H_{2(0)}}},$$

$$\Pi = \frac{p - p_a}{\rho g(H_{1(0)} - H_{2(0)}}. \tag{8}$$

Eqs. (5), (6) and (7) take the dimensionless form

$$\eta^3 - \xi^3 = 1 + 3\xi_0\,(1 - \xi_0)\,. \tag{9}$$

$$\frac{d^2\xi}{d\tau^2} = \frac{\eta}{\xi} + \frac{1}{2}\left(\frac{d\xi}{d\tau}\right)^2\left(\frac{1}{\eta} + \frac{\xi}{\eta^2} + \frac{\xi^2}{\eta^3} - \frac{3}{\xi}\right). \tag{10}$$

$$\Pi = \tilde{z} - \xi + \frac{1}{2}\left(\frac{d\xi}{d\tau}\right)^2\left(1 - \frac{\xi^4}{\tilde{z}^4}\right) + \frac{1}{3}\frac{d^2\xi^3}{d\tau^2}\left(\frac{1}{\xi} - \frac{1}{\tilde{z}}\right). \tag{11}$$

with

$$\xi = \xi_0, \quad \frac{d\xi}{d\tau} = 0 \quad \text{at} \quad z = 0 \tag{12}$$

The solution which is computed numerically, depend on the single dimensionless parameter

$$\xi_0 = \frac{H_{2(0)}}{H_{1(0)} - H_{2(0)}}. \tag{13}$$

Equation (10) reduces to the corrected Paterson's equation for a conical tube given in[3] when η is set to 1.

3. Numerical Results

After eliminating η with the help of Eq. (9), the remaining second order ordinary differential equation (Eq. 10) was simply broken in a system of two first-order differential equations and conveniently integrated numerically with the method described in.[7]

Three non–dimensional cases were analyzed here, a) $\xi_0 = 0.82$, $\eta_0 = 1.82$, b) $\xi_0 = 1.16$, $\eta_0 = 2.16$, and where $\xi_0 = 2$ and $\eta_0 = 3$ which corresponds to the non–dimensional value of the interfaces position reported in §4, its important to notice that all of these cases have a constant $\Delta H = 1.0$.

Interfaces positions, velocities and accelerations can be conveniently compared with the Uniformly Accelerated Movement equations, which ones, with the dimensionless parameters introduced in §2 take the form: $\tilde{z} = \tau^2/2$, $\dot{\tilde{z}} = \tau$, $\ddot{\tilde{z}} = 1$.

The time evolution of the free interfaces velocities (plotted in Fig. 2) clearly shows that at early stages of the movement, the upper free surface velocity moves faster than free fall, and in opposite way the lower free surface begins to move slower than free fall, but at later stages of the movement the change of direction of the velocity curves indicates that upper interface begins to decrease its velocity and lower interface raises.

In Fig. 3 the accelerations of the upper free interfaces (for the three selected cases) were plotted as a function of time and indicates that the instantaneous accelerations reached by the upper free surface decreased to values below the free fall to finally accelerate at later stages of the movement. In other hand, in Fig.4 the lower interfaces begin the movement sub–accelerated and during the development of the phenomena, increase its acceleration at values above the free fall for finally decelerate to reach the free fall acceleration. In Fig. 5 the expression for the non–dimensional dynamic pressure in-

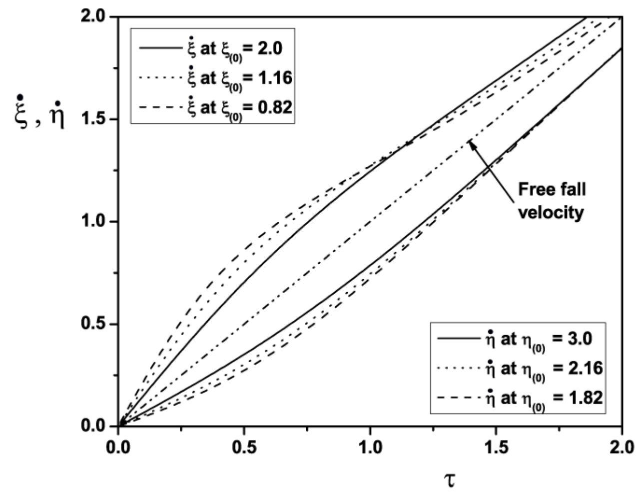

Fig. 2. Evolution of the interfaces velocity (for the selected cases $\dot{\xi}$ and $\dot{\eta}$) compared with free fall velocity.

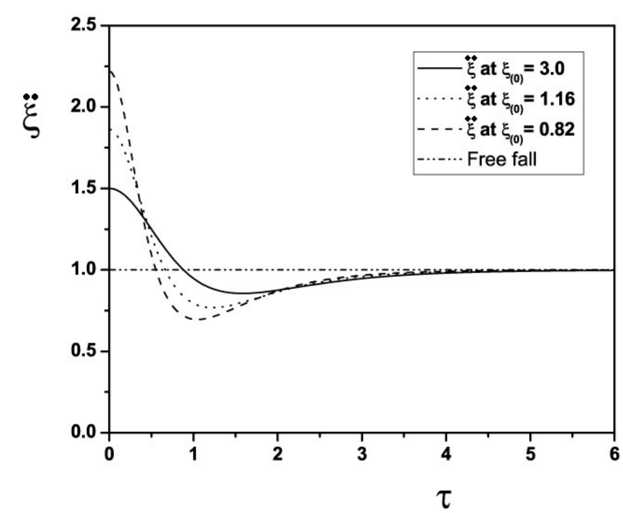

Fig. 3. Dimensionless plot of the acceleration of the upper free surface (for three selected cases) as function of time.

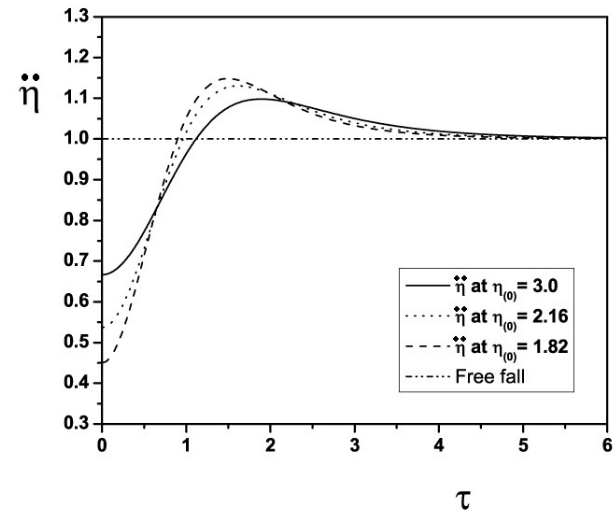

Fig. 4. Dimensionless plot of the acceleration of the lower free surface (for three selected cases) as function of time.

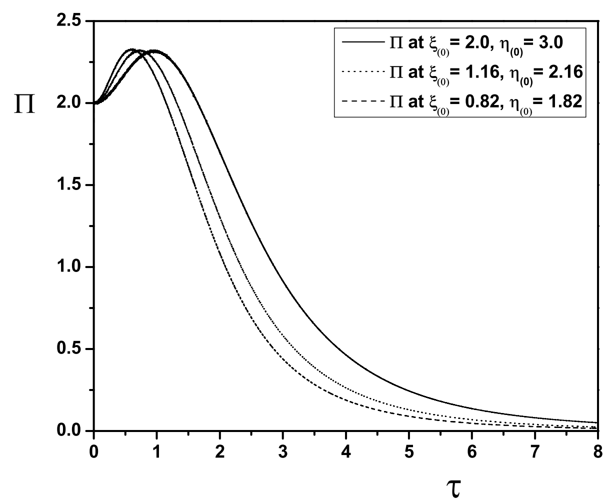

Fig. 5. Dimensionless plot of the dynamic pressure evolution of the liquid layer (for three selected cases).

troduced in Eq. 7 was plotted as a function of time for the same selected cases introduced in § 3. It is important to notice that the hydrostatic pressure (which is defined as a liquid height) is given by $\xi - \eta = 1$, but when the liquid layer leaves the rest at $\tau \gg 0$, this static pressure converts instantly in a dynamic pressure due mainly to the influence of the last term in Eq. 11.

4. Experiments

Several qualitative experiments were conducted with a glass conical tube of small angle of cone $\theta = 3°$, which is 0.48 m long, with upper diameter $D_2 = 0.03$ m and $D_1 = 0.053$ m lower diameter. The working fluid was ethanol, density $\rho = 810\,\mathrm{kg/m^3}$ and kinematic viscosity $\nu = 1.52{\times}10^{6}\,\mathrm{m^2/s}$. The value of the gravity acceleration at Azcapotzalco (Mexico City) is $g = 9.779\,\mathrm{m/s^2}$. The tube was mounted over a drilled acrylic sheet and fixed with silicon

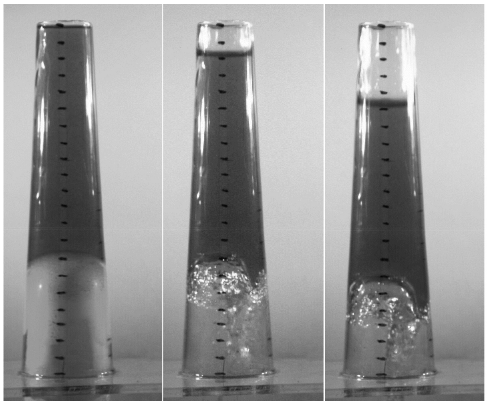

Fig. 6. Snapshots of development of the experiment, darkest zones represents the liquid slice of ethanol, and the grey region is the balloon film. Times from left to right are t = 0/500, 42/500, 59/500 s respectively.

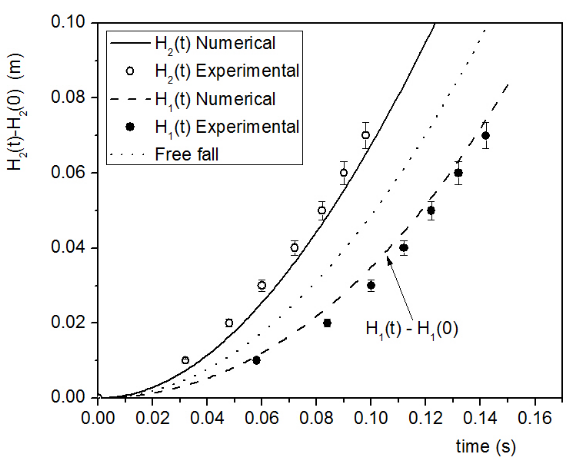

Fig. 7. Plot of the numerical computations for the position of upper interface $H_2(t) - H_2(0)$. It corresponds to $H_2(0) = 0.28$ m (solid line) and $H_1(0) = 0.42$ (dashed line) and is compared with its respective experimental values and 5% error bars are also plotted. The dotted curve indicates free fall.

cement. For the series of experiments conducted here, the liquid layer had a height of $\Delta H = 0.14$ m, the upper interface was located at a distance $H_2 = 0.28$ m from the apex of the cone and the lower interface at a distance $H_1 = 0.42$ m. An air inflated balloon was placed below the liquid to hold it in place, and the experiment was initiated by tearing the balloon with a sharp cutter. The initial lower surface is slightly deformed due to the presence of the balloon (see Fig. 6), also it is noticed a rubber film adhered to the wall of the pipe during the rupture of the balloon. All this makes a little difficult to observe the evolution of the lower liquid surface, but the evolution of the upper surface can be clearly recorded. To this end, the conical wall of the tube was marked each 0.01 m and the movement of each interface was recorded with a high speed camera Red Lake model HG-100K/HG-LE at a rate of 500 fps. Figure 6 shows a sample sequence of recorded images. A commercial software was used to extract the position of the surfaces from the images and compare them with the numerical solution. Figure 6 shows the results of the experiments and the numerical computations.

5. Conclusions

The problem of a liquid falling down an inverted cone is revisited. The cone of a finite volume of liquid in a semi-infinite tube is considered. The equation originally derived by Paterson,[1] corrected as in Torres *et al.*[3] and particularized for a conical tube, is extended to take into account the boundary condition of this case and deal with the two liquid surfaces present in the tube. The approximations if inviscid and quasi-unidirectional flow are used. The problem is shown to depend on a single dimensionless parameter which is the inverse of the ratio of the length of the tube initially occupied by the liquid to its distance to the vertex of the cone. The evolution of the upper and the lower surfaces is compared for different values of this parameter, and the periods of super acceleration of each surface are identified. New experiments have been carried out with ethanol in a conical tube with an opening angle of 3°. The results are in good agreement with the numerical computation and show the formation of the peculiar nipple observed earlier at the upper surface of the liquid.

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1) A. R. Paterson, *A first course in fluid dynamics*, (Cambridge, 1983).
2) E. Villaermaux and Y. Pomeau, J. Fluid Mech. **642**, 147 (2010).
3) A. Torres, A. Medina, F. Highera and P. Weidman, J. Fluid Mech. **723**, 653 (2013).
4) C. Treviño, S. Peralta, A. Torres and A. Medina, Europhys. Lett. **112**, 14002 (2015).
5) M. Vollmer and K.-P. Möllmann, The Physics Teacher. **51** 400 (2013).
6) E. G. Virga, Proc. R. Soc. London, Ser A., 2015, Vol. 471, p. 20140657.
7) S. Gill, Math. Proc. Cambridge Phil. Soc., 1951, Vol. 47, p. 96.