Super Free Fall of a Liquid Frustum in a Semi-infinite Cone

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Abstract. In this paper we have analyzed theoretically the super free fall of a near inviscid mass of liquid, which fills partially a small section of a very long vertical conical pipe. Through the use of a one-dimensional inviscid model, we describe the simultaneous and pecular motion of the two interphases of the liquid.

Keywords: Flow in quasi-one-dimensional system · Flows in pipes and nozzles · Navier-stokes equations

1 Introduction

Recently, it has been shown that the upper free surface of a liquid column filling a cylindrical pipe of short length, but increasing radius, reaches super free fall [1–3]. In fact, it was demonstrated that when a liquid column, in a slowly expanding conical pipe is suddenly released from the rest, by opening abruptly its bottom exit and all liquid is exhausted from the tube, the upper free surface reaches initially a super gravitational acceleration, then the acceleration becomes sub-gravitational, and finally it turns back to terminate at exactly gravitational acceleration as required [2]. In the case of pipes with a sudden expansion (interconnected pipes at different radii) the upper free surface can achieve persistent accelerations several times larger than the gravity acceleration g, and the acceleration is larger for smaller levels of filling in the upper pipe [3]. All the previous rich dynamics in confined systems contrast with those where an initially confined mass of liquid is suddenly released to the ambient due to the explosion

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of a water-filled rubber-balloon [4]. Thus, such a mass of liquid falls initially, and thereafter, it disintegrates into smaller droplets. Incidentally, the super free fall in liquids also recalls the purely-mechanical problem of the super free fall of the tip of chains falling under the gravity action in two main configurations: a vertically hanging chain released from rest and an horizontally folded chain [5]. In this work we analyze theoretically, based on a slender slope approximation, the problem of the super free fall of a specific volume of low viscosity liquid contained in a conical pipe of very large length which is supposed to be released from the rest at any part of it. Since the mass of liquid never leaves the pipe, we conceptualize such configuration as a liquid slice in a semi infinite cone. This simple system allows us to predict the dynamic behavior of both upper and lower free interfaces, during the overall history of the flow (while the liquid slice does not desintegrates due to their extreme thinness). In the last part of this communication we discuss the realization of a qualitative experiment to visualize the dynamic evolution of the liquid slice. Finally, we give the main conclusions.

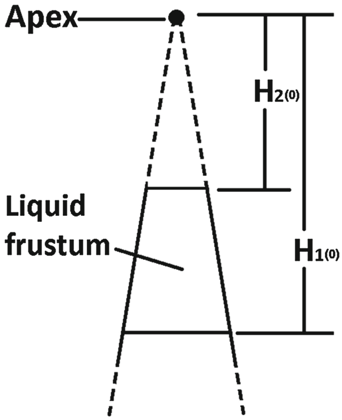

Fig. 1. Scheme of an idealized semi-infinite cone. The liquid frustum is bounded by $H_{1(0)}$ and $H_{2(0)}$.

2 Theory

Ideally, a semi-infinite conical pipe model has an infinite length and consequently, in any part of this pipe and under this configuration the radius is always increasing downwards along the flow direction. Here, we consider a finite volume of quiescent liquid confined at any part of the semi-infinite vertical cone by conforming a conical frustum (*i.e.,* a cone initially sliced by two horizontal parallel planes), both interfaces of the frustum are opened to the atmosphere at $z = H_{2(0)}$ and $z = H_{1(0)}$, respectively. Subindex (0) indicates initial positions. A scheme of the problem can be seen in Fig. 1. These two distances are taken in reference to the apex of the cone. This quasi-unidirectional model of motion of a liquid volume contained in a semi-infinite tube obeys the mass and momentum conservation equations.

$$\frac{1}{z^2}\frac{\partial}{\partial z}(z^2 u) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z}\right) = \frac{\partial P}{\partial z} + \rho g, \tag{2}$$

where z is the vertical distance measured downward from the apex of the cone, t is the time, u and P are the velocity and pressure of the liquid, ρ is the density of the liquid, g is the gravitational constant, and viscous effects have been neglected. Let $H_1(t)$ and $H_2(t)$ denote the position of the lower and upper surfaces at anytime during the movement. Then $u = dH_1/dt$, $P = P_a$ at $z = H_1$, and $u = dH_2/dt$, $P = P_a$ at $z = H_2$,

$$u = 0\,,\;\; H_1 = H_{1(0)}\,,\;\; H_2 = H_{2(0)} \;\text{ at }\; t = 0, \tag{3}$$

where P_a is the pressure outside the liquid and surface tensions have been neglected.

Equation (1) can be immediately integrated to give

$$u = \frac{dH_1}{dt}\frac{H_1^2}{z^2} = \frac{dH_2}{dt}\frac{H_2^2}{z^2}. \tag{4}$$

From the second equality, after using the initial condition (3), we obtain

$$H_1^3 - H_2^3 = H_{1(0)}^3 - H_{2(0)}^3, \tag{5}$$

which expresses the condition of conservation of the liquid volume.

Carrying equation (4) into the momentum equation (2), integrating the resulting equation from $z = H_1$ to $z = H_2$ and using the boundary conditions (3) for the pressure, we find, after some algebra

$$\frac{d^2 H_2}{dt^2} = \frac{H_1}{h_2}g + \frac{1}{2}\left(\frac{dH_2}{dt}\right)^2\left(\frac{1}{H_1} + \frac{H_2}{H_1^2} + \frac{H_2^2}{H_1^2} - \frac{3}{H_2}\right), \tag{6}$$

which is obtained by integrating (2) between $z = H_2$ to a generic value of z.

Introducing the dimensionless variables

$$\xi = \frac{H_2}{H_{1(0)} - H_{2(0)}}\,,\quad \eta = \frac{H_1}{H_{1(0)} - H_{2(0)}}\,,\quad \tau = t\sqrt{\frac{g}{H_{1(0)} - H_{2(0)}}}. \tag{7}$$

Equations (5) and (6) take the dimensionless form:

$$\eta^3 - \xi^3 = 1 + 3\xi_0\left(1 - \xi_0\right)\,, \tag{8}$$

$$\frac{d^2\xi}{d\tau^2} = \frac{\eta}{\xi} + \frac{1}{2}\left(\frac{d\xi}{d\tau}\right)^2\left(\frac{1}{\eta} + \frac{\xi}{\eta^2} + \frac{\xi^2}{\eta^3} - \frac{3}{\xi}\right). \tag{9}$$

with

$$\xi = \xi_0\,,\quad \frac{d\xi}{d\tau} = 0 \quad\text{at}\quad z = 0. \tag{10}$$

Now this final solution depends on the single dimensionless parameter

$$\xi_0 = \frac{H_{2(0)}}{H_{1(0)} - H_{2(0)}}. \tag{11}$$

3 Numerical Procedure

To obtain a numerical solution, after eliminating η by using Eq. (8), the resulting nonlinear second order equation (9) can be simply broken in a set of two stiff ordinary differential equations which will be numerically integrated by using Gill's method [7]. This method was developed from the general theory given by Kutta [8] and was chosen for this work because it is capable to reach fourth-order accuracy with the use of minimum storage registers. As mentioned in Blum [8] the two advantages of implementing Gill's method are: first, it only requires $3n + B$ storage registers whereas the standard Runge-Kutta method requires $4n+B$, where n refers to the number of coupled first-order differential equations and B is a constant; second, under Gill's method scheme the computation can be arranged and the rounding errors can be reduced significantly.

The celebrated subroutine introduced by White [6] was rewritten into a convenient from under the Fortran®95 standard, and the resulting project was compiled with the Absoft Pro Fortran®16.0.2 which is suitable to handle the proper irrational constants of Gill's method *e.g.,* $A = \sqrt{\frac{1}{2}} = 1.7071067811865475244$ with an explicit length declaration.

In order to estimate the development of the interphase acceleration we have chosen the next ΔH value at $\tau = 0$: $\xi(0) = 0.2$ and $\eta(0) = 1.2$. Those initial conditions were measured from the apex to every single interface conforming the height of the liquid frustum.

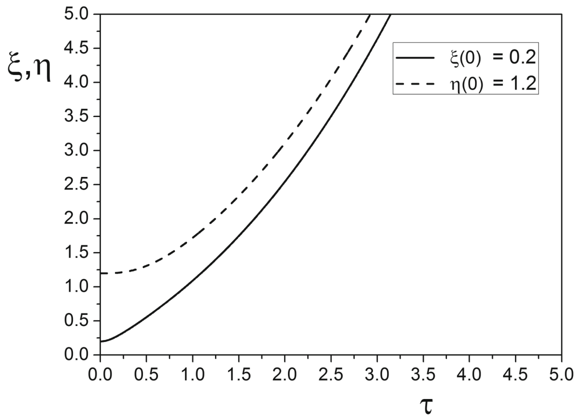

Fig. 2. Spatial evolution of both non-dimensional upper and lower free surfaces at initial conditions of filling $\xi_0 = 0.2$ and $\eta_0 = 1.2$.

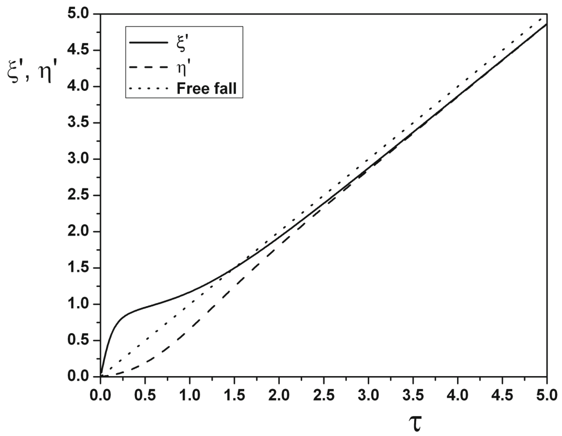

Fig. 3. Evolution of both upper and lower free interfaces $\dot{\xi}$ and $\dot{\eta}$ as a function of time.

As seen in Fig. 2 each free interface presents a similar behavior when this liquid column is suddenly released from the rest; it is apparent that the distance between interfaces decreases until a minimum distance is reached. According to Figs. 3 and 4 it is possible to conclude that at the beginning of the movement the upper free surface ξ starts to move faster than η and at later stages of the movement, both surfaces ξ and η will reach the pure free fall acceleration.

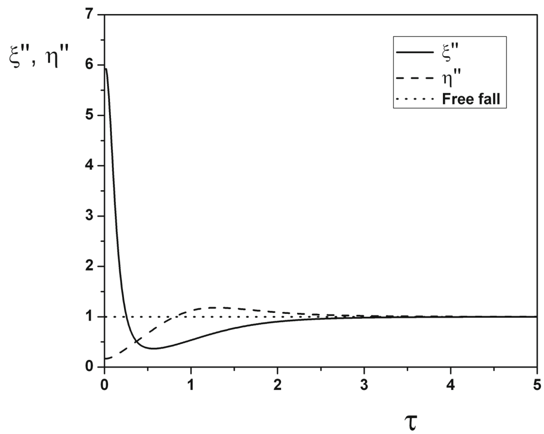

Fig. 4. Evolution of both, upper and lower interface acceleration as a function of time at initial conditions of filling $\xi_0 = 0.2$ and $\eta_0 = 1.2$.

4 Conclusions

The theoretical model based on the slender slope theory presented here, predicts the behavior of the free interfaces that conform a liquid frustum, which ideally lies in any part of a semi infinite cone and its suddenly released from the rest. The method used for the numerical computations reported here was originally designed in order to use efficiently every single storage space of the memory unit of the machine.

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